Time Preference: Experiments and Foundations*

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July 4, 2010

Abstract

In order to elicit discount functions, experiments commonly rely on separability and curvature assumptions on the underlying discounted utility representation. This paper presents an experimental procedure for elicitation that dispenses with both sets of assumptions. We posit that the behavioral meaning of discounting lies in behavior obtained by fixing the money dimension and varying only the time dimension. Based on this, we prove a general theorem for regular preferences (monotone weak orders) over dated rewards that characterizes the mapping from preferences to compatible discount functions and utility indices.

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1. Introduction

A sizable experimental literature on time preference has sought to explore and understand the structure of time preferences. This paper presents a practical procedure which makes it possible to draw conclusions about time preference under fewer assumptions than those used in the experimental literature. We first review the approach used in the literature and provide motivation for this paper.

Previous Literature. The literature of particular relevance to this paper is that on the elicitation of discount functions (as opposed to that which tests specific theories). Most experiments in the literature collect data on subjects' preferences $\succsim$ over dated rewards.[1] They typically presume that $\succsim$ admits a *Separable Discounted Utility* (SDU) representation,

$$U(m,t) = D(t) \cdot u(m), \tag{1.1}$$

of which exponential discounting $D(t) = \delta^t$ and hyperbolic discounting $D(t) = \frac{1}{1+\alpha t}$, $\alpha > 0$, are special cases. The specific data the studies seek is that on indifference points of the form:[2]

$$(x,0) \sim (y,t).$$

That is, studies either fix y and map out how the 'present value' x changes as a function of t, or they fix x and map out how the 'future value' y changes as a function of t. Since the SDU representation implies

$$D(t) = \frac{u(x)}{u(y)}, \tag{1.2}$$

the discount function D can be derived via identifying assumptions on u.

A common assumption is that u is linear (see Fredrick et al [9] for a review of the classic literature, which is further developed by Coller and Williams [6] and Harrison et al [10]). The justification is based on the idea that u is approximately linear over small intervals, and the rewards used in experiments are 'sufficiently small'. However, these experiments typically find implausibly high discount rates;

[1] A dated reward is a money-time pair (m,t) that specifies a reward $\$m$ to be received at time t.

[2] Most experiments maintain some front end delay (Coller and Williams [6], Harrison et al [10]), so that the earlier reward is obtained with some delay.

rates in excess of 100% are common. Motivated by this, Andersen et al [3] relax the linearity assumption. They use risk preferences to estimate several specifications of concave u and use these to estimate various specifications of D.

Motivation. Present/future value data captures how subjects trade-off money and time – it specifies how a present/future value (the money dimension) changes with t (the time dimension) – and because it comes from varying both the money and time dimensions, it encodes the *joint affect* of discounting and the curvature of utility. Consequently, assumptions on curvature are necessary to identify a discount function from present/future value data. This is evident from (1.2). However it is also evident that, with the SDU assumption maintained, the estimated discount function will change with the assumption the analyst adopts for u, even though the preference admits at most one SDU representation (which is unique upto a power transformation).

The extent of potential misidentification can be substantial. Experiments often fix some $\overline{x}$ and derive its future value $y(t)$ for various t,

$$(\overline{x}, 0) \sim (y(t), t),$$

as this suffices to estimate the discount function. But it can be verified that regardless of the u the analyst adopts and the D he subsquently fits, the subject can *always* be assumed to be entirely standard without loss of generality. That is, there always exists some u^* for which the corresponding D^* is exponential.[3] In particular, the analyst may conclude that (say) with the linear u assumption the subject is hyperbolic, even though the subject's behavior is in accordance with the exponential discounting model.

[3] The proof is as follows: Suppose u, D fit the data $y(\cdot)$ in the sense that $u(\overline{x}) = D(t)u(y(t))$ for all t. Take any $\delta \in (0,1)$ and consider the increasing transformation $w(z) := \delta^{-D^{-1}(\frac{u(\overline{x})}{z})}$. This is well-defined if $D(t)$ is 'well-behaved' (continuous, strictly decreasing, $D(0) = 1$ and $\lim_{t\to\infty} D(t) = 0$). Define a utility for money $v(m) = w(u(m))$, and observe that

$$v(y(t)) = w(u(y(t))) = w(\frac{u(\overline{x})}{D(t)}) = \delta^{-t} = \delta^{-t} \cdot v(\overline{x}),$$

where the last equality follows from $v(\overline{x}) = \delta^{-D^{-1}(\frac{u(\overline{x})}{u(\overline{x})})} = \delta^{D^{-1}(1)} = \delta^0 = 1$. Thus, the exponential discounting model $\delta^t \cdot v(m)$ fits the data $y(\cdot)$ in the sense that $v(\overline{x}) = \delta^t v(y(t))$ for all t, as was to be shown.

A similar proof works if the analyst elicits present value data $x(\cdot)$ for a fixed future reward $\overline{y}$.

This Paper. We seek a method of eliciting discount functions that is more robust to the possibility of misidentification. We proceed by asking two questions: (a) What is the *behavioral definition* of a discount function? (b) How can the discount functions consistent with this behavioral definition be identified?

We hypothesize that the behavior that purely and directly reveals discounting must be that obtained by fixing the money dimension and varying *only* the time dimension. Suppose that for any reward s, larger reward l and any time t, there is a time delay $\Phi_{s,l}(t)$ for which the subject exhibits:

$$(s, t) \sim (l, \Phi_{s,l}(t)), \tag{1.3}$$

that is, s at time t is as good as l at time $\Phi_{s,l}(t)$. The function $\Phi_{s,l}(\cdot)$ reveals that the 'loss of attractiveness' (discounting) of s due to a delay over the interval $[0, t]$ is equal to that of l over $[\Phi_{s,l}(0), \Phi_{s,l}(t)]$. Thus the function Φ tells us something about the agent's discount function. We posit that the behavioral meaning of discount functions is precisely Φ.

Assuming only the usual regularity assumptions (completeness, transitivity, continuity, monotonicity), our main result identifies a mapping between Φ and the set of all discount functions of the form $D(m, t)$ that are 'attributable' to the subject. Not only does this substantially relax the SDU assumption by accounting for discount functions that depend on the size of the reward, it also does not require any assumptions on u. In fact, we obtain the subject's corresponding u as part of our result.[4]

An experimental procedure based on our result identifies discount functions in a direct manner, and this suggests that possible misindentification may be less severe. Any pair of discounting theories are distinguishable from each other with finite data. For instance, both hyperbolic and exponential discounting imply linearity, $\Phi_{s,l}(t) = a_{sl}t + b_{sl}$, but exponential discounting is equivalent to the restriction $a_{sl} = 1$ whereas hyperbolic discounting requires $a_{sl} > 1$. Thus, even with finite data on Φ, substantially different discounting theories can be readily separated.

An experimental procedure that elicits D and u under weaker assumptions than those used in the literature has obvious value in terms of scientific rigor, but it can also potentially change previous conclusions. This may in turn have relevance for other fields that derive motivation from experimental results. Consider

[4]Once functional forms for D and u have been ascertained, the analyst can then proceed to estimate the parameters in the functional forms using any of the methods already used in the literature, such as via risk preferences [3].

the case of hyperbolic discounting. Early experiments sought to evaluate the standard exponential discounting model by testing its key behavioral implication, the Stationarity axiom [8], and they found *preference reversals.*[5] Preference reversals are consistent with a range of possible discount functions. The job of selecting the 'right' discount function is done by studies that elicit discount functions. These studies converged on a clear winner, the hyperbolic discounting model (Ainslie [2], Mazur [13]). Indeed, this conclusion – along with the suggestive relationship between hyperbolic discounting and self-control problems – has had a significant influence in economics where it guided the construction of new theories of intertemporal choice, which were in turn used to understand economic facts and make policy prescriptions (see Laibson [11] and subsequent literature). However, since the experimental literature presumed the SDU class, it has only established that hyperbolic discounting is the best SDU model that fits the data. But there are also non-SDU models that both generate preference reversals and are different enough that they do not carry any connotation of self-control problems.[6] Since our elicitation procedure does not presume the SDU model, it can select between the SDU and non-SDU models that span the potentially different psychological explanations for the behavior.

The remainder of the paper proceeds as follows. Section 2 presents the main theoretical results. Section 3 discusses considerations related to practical implementation. Section 4 studies a hypothetical example. Section 5 provides a comparison to the experimental literature while Section 6 clarifies the connection with the axiomatic literature. Section 7 concludes. All proofs are contained in appendices.

2. Theoretical Foundations

We present here the main theoretical results that provide the foundations for our experimental procedure.

[5] Stationarity states that how an agent trades-off money over a time interval should be independent of how far the time interval is from today. Preference reversals arise when subjects are more patient in their money-time trade-offs when the interval is farther from the present.

[6] Preference reversals are exhibited by any nonexponential separable discounting model where $\frac{D(t+1)}{D(t)}$ is increasing in t. They are also exhibited by exponential nonseparable models with $D(m,t) = \delta(m)^t$, where $\delta(m)$ increases in the size of the reward m [14].

2.1. Primitives

Assume that time is continuous and given by $\mathcal{T} = \mathbb{R}_+$, with generic elements t, t', and that the set of monetary rewards is a bounded interval $\mathcal{M} = [0, \overline{m}]$ with generic elements m, m', s, l. (In the appendix we consider a more general set $\mathcal{M}$). The primitives we consider is a revealed preference relation $\succsim$ over the set of *dated rewards* $X = \mathcal{M} \times \mathcal{T}$.

Data on such a preference forms the basis for the majority of experiments on time preference, presumably because it is the minimal and simplest data that may be used to study the basic structure of time preference while avoiding confounds with orthogonal psychological considerations. For instance, if one were to consider preferences over consumption streams, then a taste for increasing consumption patterns (which is found in experiments) may counteract discounting.

2.2. Assumptions

The only assumption we make is that the preference $\succsim$ over X is *regular* in the sense that it satisfies the following basic restrictions:[7]

1- **Order**: $\succsim$ is complete and transitive.

2- **Continuity**: For each (m, t), the sets $\{(m', t') : (m', t') \succsim (m, t)\}$ and $\{(m', t') : (m, t) \succsim (m', t')\}$ are closed.

3- **Impatience**:

(i) For all $m > 0$ and $t < t'$, $(0, t) \sim (0, t')$ and $(m, t) \succ (m, t')$.

(ii) For each m, m' such that $m' > m > 0$, there is t such that $(m, 0) \succ (m', t)$.

4- **Monotonicity**: For all t, if $m < m'$ then $(m', t) \succ (m, t)$.

Order and Continuity are standard. Monotonicity states that 'more is better'. Impatience states that 'earlier is better' and that the timing of a \$0 reward is a matter of indifference for the agent. Moreover, an immediate reward m can always be made more attractive relative to a delayed reward m' if the latter is delayed enough. Of these restrictions, transitivity is perhaps the most contentious empirically. But, as indicated in the Introduction, the results in this paper are based on the intuition that properties of discounting are revealed in behavior obtained by fixing the money dimension and varying only the time dimension, and this intuition does not hinge on transitivity or even regularity. Consequently

[7] Continuity presumes that $\mathbb{R}_+$ has the euclidean topology, any subset of $\mathbb{R}_+$ has the subspace topology, and X has the product topology.

there is reason to expect that the results in this paper may extend to non-regular preferences (such as Ok and Masatlioglu [15]).

It is straightforward to show that assuming regularity of preference is equivalent to assuming the existence of a *General Discounted Utility* (GDU) representation:

$$U(m,t) = D(m,t) \cdot u(m),$$

where $D : \mathcal{M} \times \mathcal{T} \to (0,1)$ is a *discount function* (a continuous, strictly decreasing function satisfying $D(m,0) = 1$ and $\lim_{t\to\infty} D(m,t) = 0$ for all $m > 0$) and $u : \mathcal{M} \to \mathbb{R}_+$ is a *utility index* (a strictly increasing, continuous function satisfying $u(0) = 0$). We will often refer to the tuple (D,u) with the GDU representation.[8]

2.3. Main Result

Consider the following behavioral object. For any $0 < m \leq \overline{m}$ and each t, define the function $\Phi : \mathcal{M} \times \mathcal{T} \to \mathcal{T}$ by the indifference:

$$(m,t) \sim (\overline{m}, \Phi(m,t)).$$

That is, $\Phi(m,t)$ is defined as the date such that m at t is just as good as $\overline{m}$ at $\Phi(m,t)$. Varying t affects the desirability of (m,t), and the variation in the desirability will be measured by $\Phi(m,\cdot)$. The simplest example is a linear Φ-function, $\Phi(m,t) = a(m)t + b(m)$.[9]

The main result in this paper identifies the set of discount functions attributable to the preference $\succsim$, and the utility index u that corresponds to each discount function.

Theorem 2.1. *Consider any regular preference $\succsim$ and its Φ-function. Then $\succsim$ admits the GDU representation (D,u) if and only if:*

(a) the discount function D takes the form:

$$D(m,t) = e^{-[g(\Phi(m,t)) - g(\Phi(m,0))]},$$

[8] Some elementary facts about the representations of regular preferences are as follows: For any utility index u there exists a unique representation U for a regular preference $\succsim$ [8]. Each representation U can be uniquely written in the form of a GDU representation (D,u): for any representation U, the utility index u in any GDU functional form is uniquely defined by $u(m) = U(m,0)$, and D is uniquely defined by $D(m,t) = \frac{U(m,t)}{u(m)}$ for all $m > 0$.

[9] See Lemma **??** in the appendix for implications of regularity for Φ.

for all $m > 0$ and t and for some continuous, strictly increasing and unbounded function g satisfying $g(0) = 0$; and

(b) the utility index u takes the form:

$$u(m) = e^{-g(\Phi(m,0))} \cdot u(\overline{m})$$

for all m and any scalar $u(\overline{m}) > 0$.

The result characterizes all the discount functions and corresponding utility indices that can be attributed to the preference $\succsim$. The functional forms involve an increasing transformation $g(\Phi(m,t))$ of Φ. Discount functions are defined in terms of the difference $g(\Phi(m,t)) - g(\Phi(m,0))$, whereas utility indices are defined in terms of $g(\Phi(m,0))$. The result reveals that obtaining a functional form for Φ is all that is necessary to obtain all the discount functions and utility indices attributable to the subject.

A key point to observe is that discount functions can be completely characterized in terms of the Φ-function. Therefore, the theorem provides formal justification for the claim that Φ is the *behavioral meaning* of discount functions. This observation reveals that Φ serves as a universal scale on the basis of which any theory of regular preferences can be tested, and from which the set of theories that explain a subject's behavior can be derived.

2.4. Outline of Proof[10]

The appendix proves a more general result, where $\mathcal{M}$ is not necessarily bounded. It considers the more general function $\Phi_{s,l}(t)$ defined by (1.3) in the Introduction, that is, the larger reward is not fixed at $\overline{m}$. The key intuition behind the proof is as follows. For any $0 < s < l$, consider the two indifference points:

$$(s,0) \sim (l, \Phi_{s,l}(0)) \text{ and } (s,t) \sim (l, \Phi_{s,l}(t)).$$

Then the loss of attractiveness (due to discounting) in (s,t) relative to $(s,0)$ must equal the loss in $(l, \Phi_{s,l}(t))$ relative to $(l, \Phi_{s,l}(0))$. This translates to the fact that any discount function D attributable to preference $\succsim$ must satisfy the equality:

$$\frac{D(s,t)}{D(s,0)} = \frac{D(l, \Phi_{s,l}(t))}{D(l, \Phi_{s,l}(0))}.$$

[10] This subsection can be skipped without loss of continuity.

By definition $D(s,0)=1$, and so this can be rewritten as:

$$D(s,t)\cdot D(l,\Phi_{s,l}(0))=D(l,\Phi_{s,l}(t)).$$

But this is a functional equation where D is the unknown function and Φ is the known function. The general functional form for D that we state in the theorem is in fact the general solution to this functional equation (for the case where $\mathcal{M}$ is bounded). The remainder of the proof verifies that for any solution D to the functional equation there exists a utility index u for which (D,u) is a GDU representation for the preference $\succsim$.

3. Practical Implementation

In this section we collect some observations on how the result of the previous section can form the basis of an experimental procedure.

3.1. Collecting Behavioral Data

The Φ-function can be obtained in practice by using the Becker-DeGroot-Marschak mechanism or by adapting the Multiple Price List (MPL) popularized by Coller and Williams [6] and Harrison et al [10].[11]

Intertemporal choice experiments usually avoid making immediate payments in order to remove any immediacy effects (Coller and Williams [6], Harrison et al [10]). Maintaining such a front-end delay implies that data on $\Phi(m,0)$ is not obtained. This data is not needed for our analysis if the analyst is concerned only with SDU models, since a preference that is represented by $D(t)u(m)$ when $t\in[0,\infty)$ is represented by $\frac{D(t)}{D(t_1)}v(m)$ for $t\in[t_1,\infty)$, that is, the discount function obtained by the analysis is unique up to multiplication by a scalar. Indeed, for the SDU case, in all that follows, time 0 can be taken as the definition of the earliest period $t>0$ considered by the analyst. However, if the analyst is interested in

[11] An MPL asks questions of the form "Do you prefer \$100 now or \$$x$ in 6 months?" where x varies over a grid $x_1,...,x_{N+1}$ of dollar amounts. The implied interest rate associated with x increases monotonically moving down the list, and the point at which the subject switches from preferring the earlier reward to the later reward determines an interval $[x_i,x_{i+1}]$ within which an indifference point '(\$100,now) $\sim$ (\$$z$,6 mth)' lies. Adapting to the current context, a 'Multiple Delay List' would ask a sequence of questions of the form "Do you prefer \$50 in 1 month or \$100 in t months?" where t varies over a range of time periods $t_1,..,t_{N+1}$ in a way that the implied interest rate decreases monotonically moving down the list.

non-SDU models, then $\Phi(m,0)$ must at the very least be 'projected' on the basis of the collected data. For instance, if $\Phi(m,\cdot)$ is linear for $t>0$, then $\Phi(m,0)$ may be taken as the y-intercept of the linear equation.

3.2. Convenient Functional Forms

A flexible class of functional forms for Φ that would serve most inquiries of interest is:

$$\Phi(m,t) = f^{-1}\left(\ a(m)\cdot f(t) + f(\Phi(m,0))\ \right) \tag{3.1}$$

where $a(\cdot)>0$ is a continuous decreasing function satisfying $a(\overline{m})=1$, and $f(\cdot)$ is a continuous, strictly increasing and unbounded function satisfying $f(0)=0$.

Then an immediate corollary of the result in the previous section is that: *for any scalars $r, u(\overline{m})>0$, the preference $\succsim$ can be attributed the discount function:*

$$D(m,t) = e^{-r\cdot a(m)\cdot f(t)}, \tag{3.2}$$

and the utility index u:

$$u(m) = e^{-r\cdot f(\Phi(m,0))}\cdot u(\overline{m}). \tag{3.3}$$

The discount function (3.2) generalizes exponential discounting to permit both a non-linear treatment of time (captured by f) and a dependence of impatience on the magnitude of rewards (captured by the dependence of the discount rate $ra(m)$ on the reward). This is a wide class of functional forms for D, which includes both exponential and hyperbolic discounting and their extensions – see the table below. An important special case of (3.1) is $a=1$, which gives rise to the separable class $D(t)=e^{-rf(t)}$.

The fact that the discount rate $ra(\cdot)$ is decreasing in the size of the reward is reminiscent of the magnitude effect (see Fredrick et al [9] for a review of the experimental literature). The reader should note that the restriction that $a(\cdot)$ is decreasing is an *ordinal* one, implied by basic regularity conditions.[12]

[12] To see this, write $\delta(m)=e^{-a(m)}$, suppose $s<l$ and observe that Monotonicity requires that $\delta(l)^{f(t)}\cdot u(l)>\delta(s)^{f(t)}\cdot u(s)$ and thus $\frac{u(l)}{u(s)}>\left(\frac{\delta(s)}{\delta(l)}\right)^{f(t)}$ for all t. If $\frac{\delta(s)}{\delta(l)}>1$, this inequality cannot hold for all t, a contradiction. Thus $\delta(s)\le\delta(l)$, and $a(\cdot)$ must be decreasing.

Φ-Function	Discount Function D	Generated by:
$\Phi(m,t) = f^{-1}(f(t) + f(\Phi_m(0)))$	$D(t) = e^{-rf(t)}$	$a = 1$
$\Phi(m,t) = t + \Phi_m(0)$	$D(t) = e^{-rt}$	$f(t) = t$, $a = 1$
$\Phi(m,t) = (1 + \alpha\Phi_m(0))t + \Phi_m(0)$	$D(t) = (1 + \alpha t)^{-1}$	$f(t) = \ln(1 + \alpha t)$, $a = 1$
$\Phi(m,t) = a(m)t + \Phi_m(0)$	$D(m,t) = e^{-ra(m)\cdot t}$	$f(t) = t$
$\Phi(m,t) = [a(m) \cdot t^\alpha + \Phi_m(0)^\alpha]^{\frac{1}{\alpha}}$	$D(m,t) = e^{-ra(m)\cdot t^\alpha}$	$f(t) = t^\alpha$
$\Phi(m,t) = \frac{[(1+\alpha t)^{a(s)}(1+\alpha\Phi_m(0))-1]}{\alpha}$	$D(m,t) = (1 + \alpha t)^{-\varphi(m)}$	$f(t) = \ln(1 + \alpha t)$

Table 1: Φ-functions and associated D.

There may be more than one pair of a and f satisfying (3.1) for a given Φ, and for each such pair a representation for the preference is obtained. Any representation contains the same behavioral information as the other, and so the usual considerations of tractability, simplicity and parsimony may go into selecting between representations. For instance, an SDU representation may be preferred by the analyst to a non-SDU one. On the other hand, different representations may provide different intuitive explanations for the behavior, which may be relevant for interpretation and applications. See the next section for further discussion.

3.3. Estimation

We have seen how to obtain functional forms for D and u attributable to the subject. These functional forms will generally contain free parameters (arising from r and the functions $a(m)$ and $f(t)$). Estimation of the free parameters can be done using any of the identifying assumptions used in the literature, which includes exploiting risk preferences [3] or preferences over streams [5]. Risk preference data along with the expected utility assumption place restrictions on the free parameters in u, all of which are in fact the parameters in D. Preferences over streams along with the assumption of additive separability produce a system of equations. That is, if preferences over streams are assumed to admit an additively separable representation with D and u:

$$U(m_0, m_1, ..) = \sum D(m_t, t)u(m_t),$$

then indifference points such as 'm_0 today and m_1 next month is as good as m_0' today and m_1' next month' will produce equations such as $u(m_0) + D(m_1, 1)u(m_1) = u(m_0') + D(m_1', 1)u(m_1')$. The free parameters can be estimated by determining values that solve equations derived this way.

4. Hypothetical Example

We discuss a hypothetical example to illustrate our procedure and the kind of conclusions that may be drawn from its results.

Suppose that the Φ-function is linear:[13]

$$\Phi(m,t) = a(m)t + b(m). \tag{4.1}$$

Regularity requires that $a(\cdot)$ and $b(\cdot)$ are decreasing, and that $a(\overline{m}) = 1$ and $b(\overline{m}) = 0$. Thus, the lines $\Phi(m,\cdot)$ are upward sloping, non-intersecting, and the curves for lower m lie strictly above those for higher m. A notable implication of the fact that $a(\cdot) \geq 1$ is the existence of *preference reversals* of the kind observed in intertemporal choice experiments: if $(s,0)$ is preferred to (l,d) and $(l,d+t^*)$ is preferred to (s,t^*) for some t^*, then $(l,d+t)$ is preferred to (s,t) for all $t^* \geq t$. This is satisfied vacuously if $a = 1$.

We first seek conditions under which there exists an SDU representation. Refer to m-independent discount functions $D(t)$ as *separable* discount functions. Applying the observations in Section 3.2 we obtain:

Proposition 4.1. *Suppose Φ has a linear form (4.1). A separable discount function $D(t)$ can be attributed if and only if there exists $\alpha \geq 0$ such that*

$$a(m) = 1 + \alpha b(m).$$

If $\alpha = 0$, then the only attributable separable discount function is exponential discounting,

$$D(t) = e^{-rt}, \qquad r > 0.$$

If $\alpha > 0$, then the only attributable separable discount function is hyperbolic discounting,

$$D(t) = (1 + \alpha t)^{-r}, \quad r > 0.$$

Thus, the test for the existence of an SDU representation is that the slopes $a(m)$ must be a linear function of the intercepts $b(m)$ in (4.1). If this condition is not satisfied, then a non-SDU representation can still be found: it is evident that (4.1) is always a special case of (3.1) with linear $f(t) = t$. Therefore we have:

[13] Data from a pilot study that we conducted using undergraduates at Boston University showed that a linear function provided the best fit (in the sense of adjusted R-squared) compared to any nonlinear function obtained from nonlinear regression.

Proposition 4.2. *The function Φ has a linear form (4.1) if and only if the general exponential discount function:*

$$D(m,t) = e^{-ra(m)\cdot t}, \quad r > 0,$$

where a is decreasing, is attributable.

Thus, when Φ is linear, general exponential discounting is always attributable. Such discounting departs from exponential discounting only due to a magnitude effect, whereby the subject is more patient toward larger rewards.

A noteworthy observation is that hyperbolic discounting is *behaviorally a special case* of the general exponential discount function. Therefore the analysis reveals that the magnitude effect is an alternative (and more general) explanation of the evidence – namely preference reversals – usually attributed to hyperbolic discounting. This is noteworthy because there is a significant difference between the two classes of discount functions in terms of implications relevant for economics. Experiments typically involve small stakes, but under hyperbolic discounting one would infer that experimental findings based on small stakes (such as preference reversals and dynamic inconsistency) hold for economically-significant stakes as well and thus warrant economic analysis. Under the magnitude effect, on the other hand, such an inference cannot be made since $a(m)$ may be constant for large m, in which case the subject is a standard exponential discounter for large stakes. The two forms of discounting are also substantially different in spirit. Hyperbolic discounting is suggestive of a self-control problem, whereas the magnitude effect is suggestive of bounded rationality: the former suggests a passion for the present [11] whereas the latter suggests that subjects pay greater attention to larger rewards [14]. Such differences may have implications for applications and assessments of welfare.

5. Comparison with the Experimental Literature

We provide a specific comparison with the experimental literature.

5.1. Linear u

The Introduction mentions the classic literature that assumes linear u in order to elicit a discount function. Our procedure delivers results without any assumption on u. But even if the analyst wants to make assumptions on u, he can make use of Theorem 2.1 to obtain a functional form for D, as restrictions on u translate into restrictions on g and thus on D. The difference from the classic literature will be that misindentification issues mentioned in the Introduction will be avoided.

5.2. Andersen, Harrison, Lau and Ruström (2008)

The contribution of Andersen et al [3] (henceforth AHLR) is that they explicitly account for the possibility that u may not be linear. Their strategy is to use risk preference data to estimate u and then future value data to estimate D.

The approach in AHLR requires the analyst to specify a functional form for both u and D. The key comparison with our procedure exists at this particular step. We show how to derive functional forms for both u and D from the data. These come as a consequence of finding a functional form that best fits Φ, and thus our procedure guides and disciplines the analyst's choice of functional forms for D and u to attribute to the subject. Once the functional forms have been determined, estimation can proceed as in AHLR.

Note that if the analyst wants to consider specific functional forms for D, then that is also possible using our procedure. In this case, the functional form that is fit to the data on Φ should correspond to the desired class of D.

5.3. Other Studies using Richer Data

Like AHLR, Takeuchi [17] makes use of risk preferences. Denote by (m, p) a lottery that pays $\$m$ with probability p and 0 otherwise, and let $\succsim_r$ represent preferences over such probabilistic rewards. The author observes that for any pair of rewards s, l such that $s < l$, there exists a probability p and a delay τ such that $(s, 1) \sim_r (l, p)$ and $(s, 0) \sim (l, \tau)$, and that under expected utility theory and the SDU model (1.1) it may be inferred that:

$$D(\tau) = \frac{u(s)}{u(l)} = p.$$

Thus the author elicits D without any assumptions on u. Observe also that $\tau = \Phi_{s,l}(0)$ in our notation.

Andreoni and Sprenger [4] consider preferences over consumption streams. They ask subjects to choose their allocation of an endowment over two periods for different interest rates, and they elicit intertemporal demand curves generated from a collection of convex intertemporal budget sets. They use this to estimate an additively separable utility function $\sum D(t)u(m_t)$ with CRRA u.

An observation about the four procedures discussed above is that they are intimately tied to the identifying assumption they exploit, whether it be expected utility or additive separability. Our procedure does not require an indentifying assumption in order to determine functional forms for discounting. Moreover, for estimating a functional form, the analyst can use any of the identifying assumptions used in the literature.

5.4. Attema, Bleichrodt, Rohde and Wakker (2009)

A paper closer in spirit to ours is that by Attema et al [5] (henceforth ABRW). In common with this paper, ABRW focus on data derived by fixing the money dimension and varying the time dimension. The key difference is that ABRW focus on the SDU model (1.1), and their derivation of D is tailored for it, while our paper is formulated more generally for preferences that satisfy only basic regularity conditions (such as transitivity and monotonicity). Another difference is that they do not seek to derive a corresponding u, but rather circumvent it.

Their procedure is as follows. For any rewards s, l such that $s < l$, find t_1 where $(s, 0) \sim (l, t_1)$. Then for $i = 2, .., n+1$, iteratively find t_i for which

$$(s, t_i) \sim (l, t_{i-1}).$$

ABRW note that under the SDU model, $\frac{D(t_{i+1})}{D(t_i)} = \frac{u(s)}{u(l)}$ for all i and thus

$$\ln D(t_1) - \ln D(0) = \ln D(t_2) - \ln D(t_1) = ... = \ln D(t_{n+1}) - \ln D(t_n).$$

One way of seeing how D can be pinned down is the following. Since $D(0) = 1$ and D is unique up to a power transformation, any arbitrary value a can be assigned to $D(t_1)$. Then $\ln D(t_1) - \ln D(0) = \ln a$, and by the displayed equalities, $\ln D(t_2) = 2 \ln a$ and so on. Consequently $\ln D$ is identified on $\{0, t_1, .., t_{n+1}\}$. Fitting a curve and transposing for D identifies the discount function D up to the parameter a.

6. Relation with the Axiomatic Literature

This paper approaches the practical question of how to design intertemporal choice experiments from the perspective of decision theory. The main result in the paper identifies and characterizes the general mapping between behavior and general discounted utility representations: the general mapping is obtained by first recognizing that the behavioral meaning of discounting must lie in Φ, and second by recognizing that functional equations provide a means of characterizing the connection between Φ and the representations (D, u). This result suggests a procedure for experimentally testing theories and eliciting discount functions. But the result is simultaneously a means of axiomatizing *any* intertemporal choice model for regular preferences – the analyst need only find axioms that imply restrictions on Φ for D has the desired properties.

The ideas used in the axiomatizations of intertemporal choice models in the literature are disparate, exploiting specific aspects of the particular model being studied. Specific functional equations are used by Loewenstein-Prelec [12] and Ok and Masatlioglu [15] to axiomatize specific separable discounting models, while Fishburn and Rubinstein [8] use a constructive procedure. These axiomatizations do indeed involve restrictions on the Φ-function, but with the exception of the axiomatization of the exponential discounting model [8], they impose restrictions like the *Thomsen Condition* [8] which are not transparently about the Φ-function. But Thm 2.1 reveals that the *exhaustive* behavioral expression of discounting lies in Φ, and thus any discounting model can be characterized solely in terms of this choice data. We demonstrate this message in Thm **??** in the appendix by axiomatizing the SDU model.

The idea of looking at properties of preference over multi-dimensional alternatives that vary only one dimension has been used in the theoretical literatures on multi-attribute utility theory and decision under risk (see Fishburn [7] and Wakker and Deneffe [18] resp., for instance). It has been discussed recently by Rohde [16] in the context of intertemporal choice to define a preference-based measure of distance from exponential discounting. However the idea is used differently, and for different purposes, in these papers compared to our paper. For example, see Attema et al [5], discussed earlier.

7. Concluding Remarks

The experimental procedure proposed in this paper has several features:

– The methodological foundations of our procedure are based on a study of behavioral foundations of discounting theories. The latter provides the general connection between behavior and desired utility constructs (discount functions and utility indices), and thus permits an analysis under very weak assumptions.

– The procedure can test hypotheses involving the conjunction of D and u. While numerous studies uncover the existence of preference reversals that support hyperbolic discounting, we can check in addition whether there are hyperbolic discounting representations with a concave u that are consistent with the agent's preferences over dated rewards.

– Earlier studies find that discount functions estimated using the linear u assumption exhibit a magnitude effect, which suggests the intuitive idea that subjects are more patient toward larger rewards. However, this evidence for the magnitude effect can be explained in terms of curvature of utility [12]. Our procedure can be used to rigorously test for discounting that is independent of the size of rewards – see Appendix **??**.

– While the procedure is defined for the analysis of time preference, it applies readily to other domains as well. For instance, experiments on risk often offer subjects lotteries that have one nonzero payoff. Such lotteries can be written as (m, p), where p is the probability of the nonzero outcome. By defining 'time' as $t = \frac{1}{p} - 1$ our procedure becomes immediately applicable to the study of risk preference, where the general representation takes the form $U(m, p) = f(m, p)u(m)$ and where f is the decision weight.

A. Appendix

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