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Choice in Tree-Based Decision Problems

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CHOICE IN TREE-BASED DECISION PROBLEMS ^{*}

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June 21, 2009

Abstract

Decision makers often have to make choices from alternatives which appear in a given structure to her. For example, Rubinstein and Salant (2006) study the case where the decision maker has to make the choice from set of alternatives which appear in the form of a list. RS have characterized such choice functions from lists with some axioms. In this paper we look at the case where decision maker has to make sequential decisions and thus alternatives appear in the structure of a tree. For instance, decisions made on agendas in legislature often follow a tree structure (Ordeshook and Schwartz (1987)). Here we characterize choice from binary trees with some plausible axioms. We show that chosen element is the maximal element according to a weak order over the set of alternatives and in case of indifference a fixed tie-breaking rule is used. For ternary and higher order trees the choice functions are either Pure Preferences based or Procedural or Single Maximal Element Preference based. Lastly we characterize choice in most general case, i.e. choice from mixed trees.

JEL CLASSIFICATION: D00, D01

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1 INTRODUCTION

Our model deals with situations where a decision maker (*henceforth DM*) faces a decision problem which has a sequential or tree structure. It is often the case that the DM encounters the alternatives in a particular structure, in the form of a list or in the form of a tree. Casual observations indicate that choice in these situations depends on the structure in which the alternatives are presented to the DM. While purchasing a product online, the alternatives are listed from left to right or top to bottom. In this case, choice depends on the list DM faces while browsing the internet. The order in which alternatives are listed might influence her choice. Similarly alternatives can appear in the form of a tree-where decisions are taken sequentially. We give some examples.

Amendment Agendas (Ordeshook and Schwartz (1987)). Such agendas work as follows: a sequence of alternatives is given, and a vote is taken between the first two, after which the winner fights with the third alternative. Then the winner faces the fourth alternative. For instance consider the motions on the floor of a senate regarding a bill. The issues are : a bill, an amendment to the bill, and an amendment to the amendment. Then there are four possible outcomes: (i) the status quo, (ii) the unchanged bill, (iii) the bill amended and (iv) the bill changed by the amended amendment. The procedure requires sequential voting where the first vote is between (iii) and (iv), i.e. whether to amend the amendment. Next vote is taken between the winner and (ii), i.e. decision regarding whether to amend the bill. Lastly the vote is taken between the winner and (i). So the last decision is whether to pass the final form of the bill.

We can represent this agenda by the tree in following figure:

Here

q : the status quo

b : unchanged bill

a : bill changed by original amendment

c : the bill changed by the amended amendment

Here the alternatives appear in the form of a tree.

We consider second example: consider DM is planning to purchase a product (e.g. air-conditioner) online. If DM visits one of the online shopping website, she has to first choose type of air-conditioner (by clicking on the hyperlink provided on the webpage) she is planning to buy, i.e whether it is a split or an window air-conditioner. Suppose DM chooses window

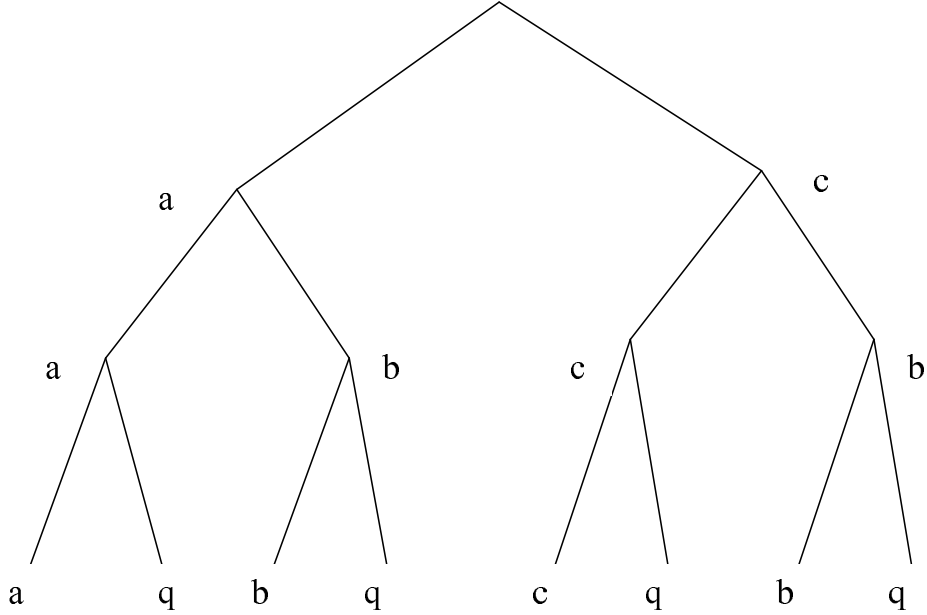


Figure 1: Tree showing choice problem in Amendment Agendas

type, then she has to choose from all available capacities of air-conditioners, i.e. whether she wants a 1 tonne capacity or 1.5 tonne capacity or even higher capacities. Once she chooses capacity the website will show all available brands, e.g. Hitachi, Career, LG, Samsung etc. We observe that this is a sequential choice and entire decision making takes the form of a tree. We illustrate it in the following figure:

So DM in above examples face the alternatives in the form of a tree. In agenda formation or in case DM is choosing to purchase a product online etc. decision making process takes such sequential structure. Thus a characterization of choice functions from tree-based decision problems seems interesting and we address this question in this paper.

Initially we consider choice functions from binary trees that satisfy some plausible axioms. We show that the chosen element is maximal according to some complete, transitive binary relation over the set of alternatives. If the maximal set is not singleton there exists a tie-breaking rule. Next we consider choice functions from ternary or trees with higher order. In this case we show choice function chooses either the maximal element from a tree according to a strict preference order over the set of alternatives (or according to a preference ordering with a single maximal element) or choice function follows a given procedure to choose the alternative: given a k -ary tree, choice function chooses a particular branch i.e. r -th ($1 \leq r \leq k$) branch from the left from each node and reaches the alternative to choose.

This work is motivated by Rubinstein and Salant (2006)(henceforth RS). They have

characterized choice functions from *lists* with the main result that the chosen element is the maximal element according to some complete, transitive binary relation. If the maximal set is not singleton then the choice is made using a fixed tie-breaking rule. It is notable that the result in RS is similar to our result in the binary tree case. However the case of choice from ternary trees or of higher order is different.

Salant and Rubinstein (2007) (henceforth SR) have considered a larger question. They have characterized choice functions when the choice is influenced by some observable information which deem irrelevant to the rational assessment. They call this information as the *frame* and the choice problem that internalizes this frame is an *extended choice problem*. SR identifies the conditions under which there exists either a transitive or transitive and complete binary relation over the alternatives such that the chosen alternative is the maximal according to that binary relation from set of alternatives. In present case, the frame of a choice problem is *tree*, since alternatives appear in a tree-form. Given the same set of alternatives if the tree changes, i.e. the frame changes, choice might also be affected.

However, in Section 2 we discuss axioms with some examples. In Section 3 we characterize choice functions from binary trees. We then consider ternary and k -ary case. In Section 5 we briefly consider choice functions in the most general case, i.e. choices from mixed trees.

2 THE MODEL

We consider a finite set of alternatives X with $|X| = n$.

Definition 1 *A rooted tree is a tree in which every vertex has either 0 or k offsprings. A vertex with 0 offsprings will be called a terminal node labeled with an element from X .*

Definition 2 *An elementary tree of order k is the tree with single vertex and k offsprings, each offspring labeled with an element of X .*

In next two figures we present elementary binary and ternary trees, followed by some definitions:

Definition 3 *A tree having equal number of choices, say k ($k = 2, 3, \dots$) at each node is called a k -ary tree.*

Trees with $k = 2$ or $k = 3$ are called binary and ternary trees respectively. In general we say that k is the order of a k -ary tree.

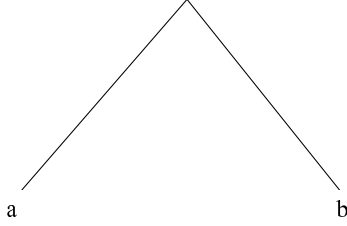


Figure 2: Elementary Binary Tree

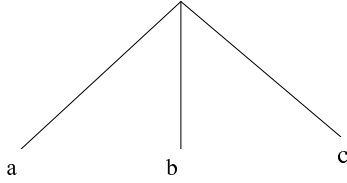


Figure 3: Elementary Ternary Tree

Definition 4 *Trees which allow different number of offsprings at different nodes is called “Mixed trees”.*

An example of the mixed tree is provided in the following figure 3:

Let Γ^k denote the set of all k -ary trees. Thus Γ^2 is the set of all binary trees. Also let the set of all possible trees be denoted by Γ . Thus $\Gamma = \cup_{k \geq 2} \Gamma^k$. For any $t \in \Gamma$, let $X(t) \subset X$ be the set of alternatives which appear as terminal nodes in t . Also $\Gamma^k(B)$ denotes the set of k -ary trees formed with elements from B , $B \subset X$. Similarly, $\Gamma(B)$ denotes the set of all possible trees formed with elements from B , $B \subset X$.

Definition 5 *We define the length of an alternative, say x , as the number of vertices in the path from origin to the terminal node. We denote it by $l(x)$.*

We now make an assumption. Consider any k -ary tree $t(k \geq 2)$, such that all alternatives in $X(t)$ do not have same length.

A tree can be represented as an ordered sequence $t \equiv (X^1, X^2, \dots, X^M)$ where each $X^j \in \underbrace{X \times X \times \dots \times X}_{r^j}$, $j = 1, 2, \dots, m$. Thus each X^j is a collection of r^j elements of X .

Note that X^j may contain repetitions.

The interpretation here is that $X^1 \cup X^2 \dots \cup X^M$ constitutes the set of terminal nodes in t . All nodes represented in $X^j(j = 1, \dots, M)$ are successors of the same non-terminal

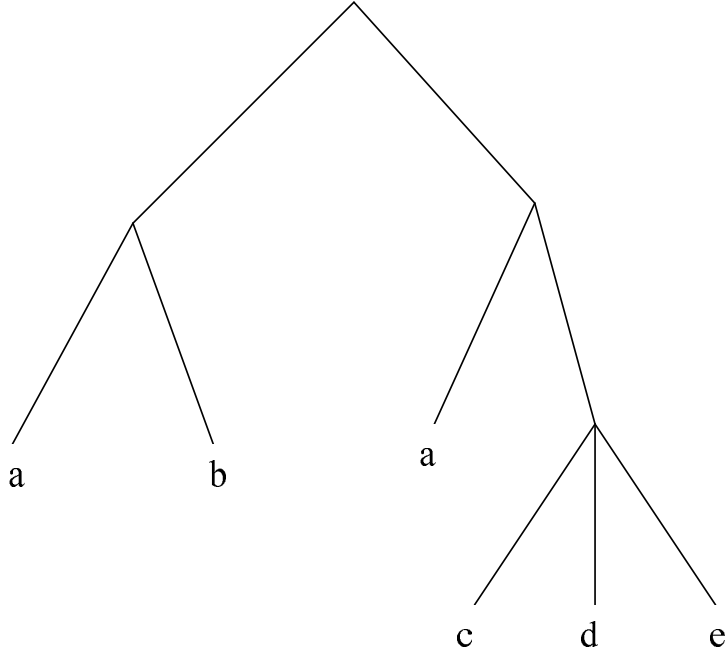


Figure 4: Mixed Trees

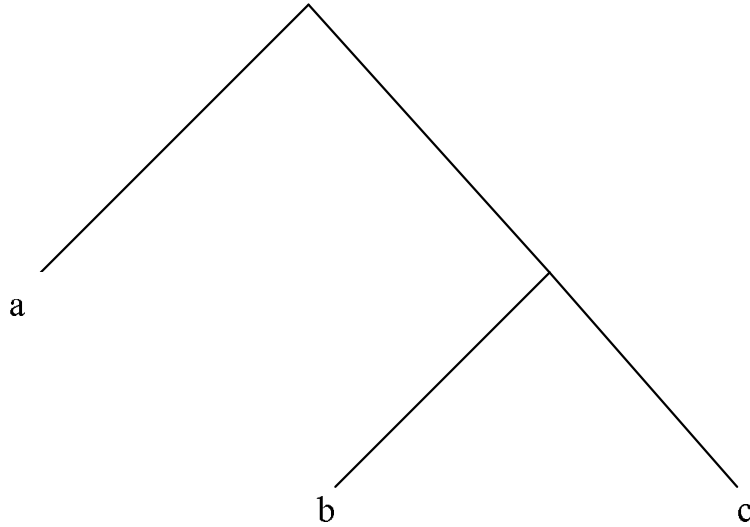


Figure 5: Illustration: Assumption 1

node. Thus consider the tree t in Figure 1. t can be equivalently represented as $t \equiv (X^1, X^2, X^3, X^4)$, where $X^1 = \{a, q\}$, $X^2 = \{b, q\}$, $X^3 = \{c, q\}$ and $X^4 = \{b, q\}$.

Note that the sets X^1, \dots, X^M are ordered. Thus we will refer to X^j as the j -th set from the left. Similarly if $X^j \equiv (x_1^j, \dots, x_k^j, \dots, x_r^j)$, then x_k^j is the k -th element from the left in

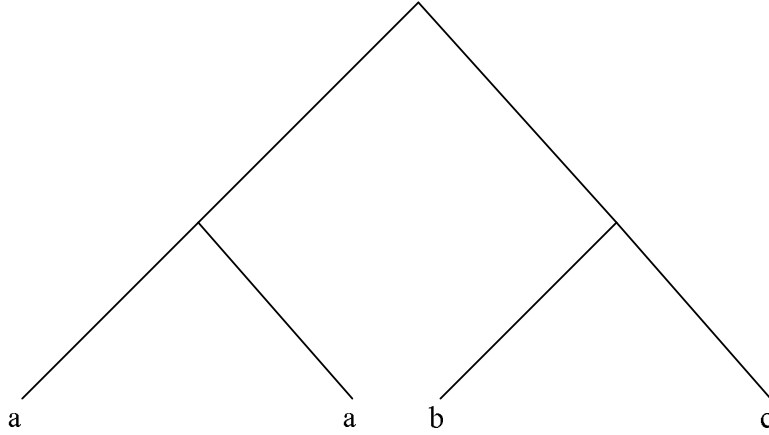


Figure 6: Illustration: Assumption 1

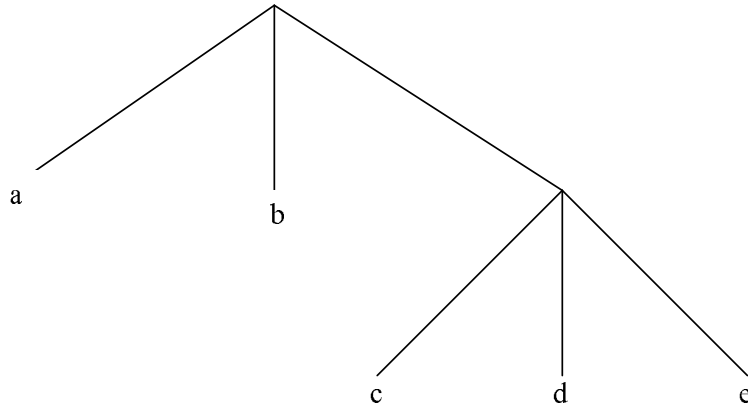


Figure 7: Illustration: Assumption 1

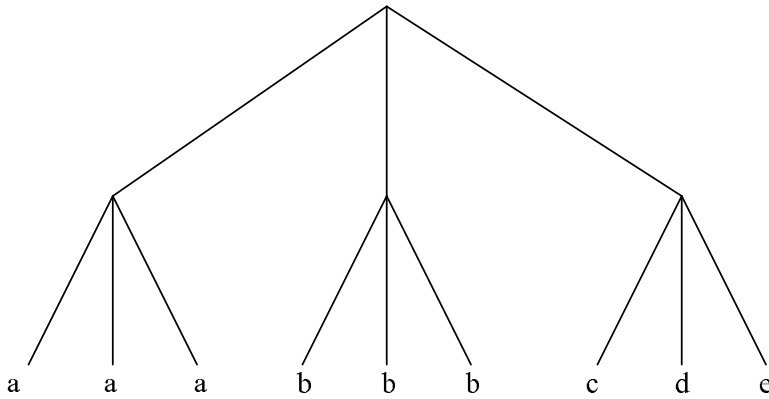


Figure 8: Illustration: Assumption 1

X^j . In general we can also say without ambiguity that x_j^k is to the left of $x_{j'}^{k'}$ if $j < j'$ or $j = j'$ and $k < k'$.

It follows from above discussion that any tree $t \equiv (X^1, X^2, \dots, X^M)$ can also be written as $t \equiv \{(x_1^1, x_2^1, \dots, x_{r_1}^1), \dots, (x_1^M, x_2^M, \dots, x_{r_M}^M)\}$ where $X_i = (x_1^i, x_2^i, \dots, x_{r_i}^i)$.

We now provide a comment regarding presentation of a k -ary tree. We have already defined *elementary trees*. In elementary trees of any order k there is one vertex and k terminal nodes labeled with elements. Thus an elementary tree of order k is a k -ary tree where there are k choices from vertex (or initial node) and successive nodes are terminal nodes. Interpretation of the mode of representation, as discussed before, for any non-elementary tree is following: for any k -ary tree $t \equiv (X^1, X^2, \dots, X^M)$, where t is not an elementary tree, X^j contains k elements, i.e. $r^j = k$, $1 \leq j \leq M$. Also since each node of the tree has equal number of offsprings $M = k^m$, $m = 1, 2, \dots$

We now have following definition:

Definition 6 A choice function C from trees is a map $C : \Gamma \rightarrow X$, such that $C(t) \in X(t)$, $\forall t \in \Gamma$.

For k -ary trees and elementary tree, we have following definitions:

Definition 7 A choice function C from k -ary trees is a map $C : \Gamma^k \rightarrow X$, such that $C(t) \in X(t)$, $\forall t \in \Gamma^k$.

For example consider the following choice rule C : for any binary tree $t \in \Gamma^2$ such that $t = (X^1, X^2, \dots, X^M)$, $C(t) = x_1^1$, i.e. first element from the left in the set X^1 is chosen. Therefore for a binary tree $t \equiv (X^1, X^2)$, where $X^1 = \{a, b\}$, $X^2 = \{c, d\}$, we have $C(t) = a$.

Definition 8 A Choice Function on elementary trees is a choice function that chooses an element from $X(t)$ for every elementary tree t .

We will consider few choice functions here.

Definition 9 A binary relation R is called **Pure Preference** if it satisfies complete, anti-symmetric and transitive.

Definition 10 A choice function is **Pure Preference based (PREF)** if there exists an complete, anti-symmetric and transitive order \succ on X such that for all trees $t \in \Gamma(B)$, $C(t) = (M(X(t), \succ))$. We have already noted that $X(t) = B \subset X$ and $M(X(t), \succ)$ is the maximal set from $X(t)$ according to order \succ .

Suppose $x(r;t)$ denotes the element in set $X(t)$ that can be reached in the tree t by picking the r -th branch at every non-terminal node, starting from the initial node. For any k -ary tree, $1 \leq r \leq k$. For any mixed tree, range for r in definition of $x(r;t)$, depends on numbers of branches emanating from different nodes in the mixed tree. For instance, for the tree in Fig. 4, r cannot be more than 2. This is because there are two branches emanating from initial node in the tree and thus r -th branch with $r \geq 3$ does not exist at the initial node. Thus $x(r;t)$ cannot be defined for $r \geq 3$.

Definition 11 A choice function C is **Purely Procedural** (*PROC*) in Γ , if either of following holds:

(i) there exists r , $1 \leq r \leq 2$ such that for all $t \in \Gamma(B)$, $B \subset X$, $C(t) = x(r;t)$.

(ii) $C(t)$ for any $t \in \Gamma(B)$, $B \subset X$ is the element that can be reached in tree t by always choosing the last branch from the left at every non-terminal node.

In case (i) holds, we denote this choice function by *PROC*(r). If (ii) holds or if $r = 1$ in (i), we call the choice function **Extremal** and denote it by *EXT*.

Remark 1 We note in above definition, range for r is limited to $1 \leq r \leq 2$, because C is defined over all trees in Γ . Therefore trees, in which only two branches are emanating from some non-terminal node, are included in the domain. Thus r cannot be more than 2 in the definition of $x(r;t)$.

The definition of *PROC*(r) for k -ary trees is following:

Definition 12 A choice function C is **Purely Procedural** (*PROC*) in Γ^k , if there exists r , $1 \leq r \leq k$ such that for all $t \in \Gamma^k(B)$, $B \subset X$, $k \geq 2$, $C(t) = x(r;t)$.

We also denote this choice function by *PROC*(r).

Remark 2 Comparing definitions of *PROC* from mixed trees and *PROC* from k -ary trees, we observe, in second case we get *EXT* when $r = 1$ or $r = k$.

We need few definitions before we define next type of choice function:

Definition 13 A binary relation is a Weak Ordering if it is complete, reflexive and transitive, i.e. for any two elements $x, y \in X$, either $x \succeq y$, or $y \succeq x$. Also if $x \succeq y$, $y \succeq z$, then $x \succeq z$, i.e. \succeq is transitive. Also if for $x \neq y$, $x \succeq y$, $y \succeq x$, then we say that x and y are indifferent.

Definition 14 Given a set of alternatives X , and a weak ordering \succeq over X , we define the Maximal Set, as $M(X, \succeq) = \{x | x \succeq y \text{ for all } y \in X\}$.

Definition 15 Given a set of alternatives X , and a weak ordering \succeq over X , we define an Indifference Set as $I_i \subset X$ such that for any $x_i, x_j \in I_i$, $x_i \succeq x_j$ and $x_j \succeq x_i$. Given X and \succeq , let us denote the set of all indifference sets by $I\{X, \succeq\}$. We note that $M(X, \succeq) \in I\{X, \succeq\}$. Also for some $X' \subset X$, if $M(X', \succeq)$ is also an indifference set.

Definition 16 Given a set of alternatives X and a weak order \succeq over X , we define Binary Indicator Function $\delta_{\{X, \succeq\}}$ as a map $\delta_{\{X, \succeq\}} : X \rightarrow \{left, right\}$ such that $\forall x, y \in I_i$ implies $\delta_{\{X, \succeq\}}(x) = \delta_{\{X, \succeq\}}(y)$ for any $I_i \in I\{X, \succeq\}$.

Remark 3 Given a set of alternatives X and weak order \succeq , for any set of alternatives $x_1, x_2, \dots, x_m \in M(X', \succeq)$, we have $\delta_{\{X, \succeq\}}(x_1) = \delta_{\{X, \succeq\}}(x_2) = \dots = \delta_{\{X, \succeq\}}(x_m)$, where $X' \subset X$. This follows from that $M(X', \succeq)$ is an indifference set.

We define a choice function:

Definition 17 A choice function $C : \Gamma \rightarrow X$ is **Mixed** if there exists a weak order \succeq over X and a binary indicator function $\delta_{X, \succeq}$, such that for any $t \in \Gamma$ we have:

(i) $C(t)$ is the left-most alternative in $M(X(t), \succeq)$ if $\delta_{\{X, \succeq\}}(x) = left$ for any $x \in M(X(t), \succeq)$; or

(ii) $C(t)$ is the right-most alternative in $M(X(t), \succeq)$ if $\delta_{\{X, \succeq\}}(x) = right$ for any $x \in M(X(t), \succeq)$.

We note above choice function is defined as **Mixed**, because a a Mixed choice function can also be *PREF* or *PROC*, depending on the weak order \succeq . If the weak order is anti-symmetric, then it is a Pure Preference and hence the Mixed choice function becomes *PREF*. Also a Mixed choice function is *PROC*, when for any $x, y \in X$, we have $x \succeq y$ and $y \succeq x$, where \succeq is the weak order over X , associated with the Mixed choice function. This is because, since $x \succeq y$ and $y \succeq x$ for any $x, y \in X$, given any tree t , the binary indicator function $\delta_{\{X, \succeq\}}$ chooses either the left-most or the right-most alternative from $X(t)$ (depending on whether $\delta_{\{X, \succeq\}}(x) = left$ or $\delta_{\{X, \succeq\}}(x) = right$), which is an indifference set.

Now we make an important assumption which will remain in force till Section 5:

Assumption 1 *DM always faces choice problem from k -ary trees.*

We will relax this assumption later on to allow for dummy choices, i.e. *Mixed Trees*. We now define following operation on a k -ary tree:

Definition 18 *Let $t_1, t_2, t_3, \dots, t_k \in \Gamma^k$, where each tree t_i can be represented as $t_i \equiv (X_i^1, X_i^2, \dots, X_i^{n_i})$ ($n_i = k^{m_i}$; m_i : any non-negative integer). Then we define the horizontal concatenation of t_1, t_2, \dots, t_k , denoted by $(t_1 \circ t_2 \circ \dots \circ t_k)$, as the k -ary tree t , such that*

$$\begin{aligned} t &\equiv (X_1^1, X_1^2, \dots, X_1^{n_1}, X_2^1, X_2^2, \dots, X_2^{n_2}, \dots, X_k^1, X_k^2, \dots, X_k^{n_k}) \\ &\equiv (X^1, X^2, \dots, X^N); \end{aligned}$$

$$\text{Here } X^1 = X_1^1, X^2 = X_1^2, \dots, X^N = X_k^{n_k}.$$

Alternatively t can also be represented as

$$t \equiv \{(x_1^1, x_2^1, \dots, x_k^1), (x_1^2, x_2^2, \dots, x_k^2), \dots, (x_1^l, x_2^l, \dots, x_k^l)\}.$$

Here l depends on k and n_i ; $i = 1, 2, \dots, k$.

Remark 4 *We note that this operation has been defined such that resulting tree is also a k -ary tree. We also note that this operation can be used recursively. For instance, suppose t', t'', t''' are three ternary trees, each formed by horizontal concatenation of 3 trees from Γ_3 . A new ternary tree t^* can be formed by horizontal concatenation of t', t'' and t''' . Thus a k -ary tree can be formed by horizontal concatenation of k trees from Γ_k where each of these k -ary trees are formed by horizontal concatenation of again k trees from Γ_k and so on. We will use this recursive property in one axiom in the next section.*

Given above definition we can represent any tree as a horizontal concatenation of elementary trees. For instance, consider the ternary tree $t \equiv (X^1, X^2, X^3)$, where $X^1 = \{x_{11}, x_{12}, x_{13}\}$, $X^2 = \{x_{21}, x_{22}, x_{23}\}$ and $X^3 = \{x_{31}, x_{32}, x_{33}\}$. This is the horizontal concatenation of elementary trees $t_1 \equiv \{x_{1i}\}$, $t_2 \equiv \{x_{2i}\}$, $t_3 \equiv \{x_{3i}\}$. Thus we can represent t as $t \equiv t_1 \circ t_2 \circ t_3$.

Now we need following definitions :

Definition 19 Consider a k -ary tree $t \equiv (X^1, X^2, \dots, X^M)$, $M = k^m$ (m : an integer). For any element x_i^j (we note that x_i^j is the i -th element from the left in X^j), we define *the Partner Set* of x_i^j as the set $\{x_l^j; l \neq i \text{ and } x_l^j \in X^j\}$. We denote it by $\{\beta(x_i^j)\}$. A representative element from the Partner Set is $\beta(x_i^j)$.

For an elementary binary tree, Partner Set of an alternative is singleton. For instance, consider the elementary binary tree $t \equiv \{a, b\}$. Here $\{\beta(a)\} = \{b\}$, $\{\beta(b)\} = \{a\}$.

In the next section we consider axioms which help us to characterize choice functions from trees.

3 AXIOMS

In this section we describe the axioms which characterize choice functions from trees. These axioms are similar in spirit to the axioms used in RS (2006).

Axiom 1 Concatenation Independence (CI): *A Choice function from trees of order k satisfies Concatenation Independence (CI) if for any arbitrary k -ary tree $t \equiv \{t_1 \circ t_2 \circ \dots \circ t_k\} \in \Gamma^k$, the following holds:*

$$\begin{aligned} C(t) &= C\{C(t_1), C(t_2), \dots, C(t_k)\} = \\ &C\{C\{C(t_1^1), C(t_1^2), \dots, C(t_1^k)\}, C\{C(t_2^1), C(t_2^2), \dots, C(t_2^k)\}, \dots, C\{C(t_i^1), C(t_i^2), \dots, C(t_i^k)\}, \\ &\dots, C\{C(t_k^1), C(t_k^2), \dots, C(t_k^k)\}\} = \dots \\ &= C\{C\{C\{\dots\}, C\{C\{\dots\}, \dots, C\{C\{\dots\}\}\}. \end{aligned}$$

Here t is a concatenation of t_1, t_2, \dots, t_k and t_i is concatenation of $t_i^1, t_i^2, \dots, t_i^k$ ($i = 1, 2, \dots, k$) and so on. Observe that CI has been defined recursively. If we partition a tree t of order k into a set of sub-trees so that former is a concatenation of these sub-trees, then CI works on these partitions inductively. It implies that the DM chooses the same element from an arbitrary tree whether he chooses from the tree as a whole, or if he partitions the tree into sub-trees, chooses from each sub-tree and then again chooses from the reduced form tree and so on.

Above axiom can be related to *Partition Independence (PI)* axiom in RS (2006). A choice function satisfying PI chooses the same alternative from a list whether he chooses from the whole list or he partitions the list into sub-partitions and chooses from the list comprising chosen alternatives from the sub-partitions. Here CI implies the same choice from a tree as a whole or from a tree comprising choices from sub-trees which are concatenated to form the whole tree. Similarly we can relate this axiom to several *Path Independence* axioms (e.g. Plott (1973)) used in social choice theory.

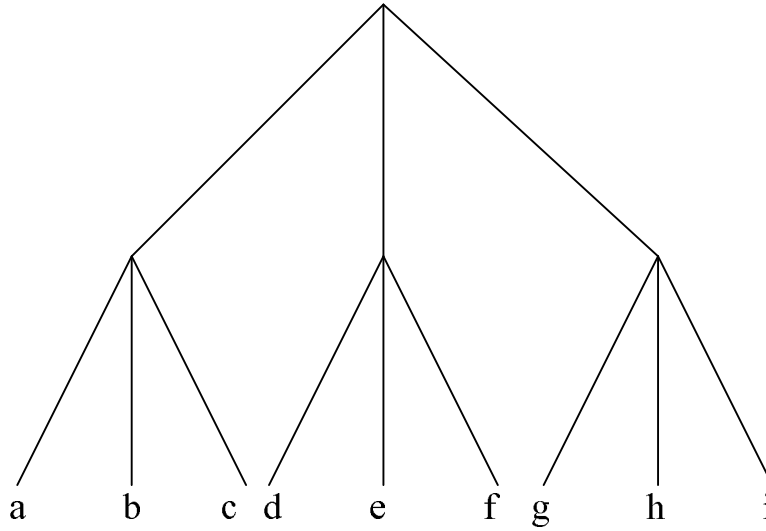


Figure 9: A ternary tree t

Example 1 Consider choice from following tree t :

Consider choice from trees t_1 , t_2 and t_3 , such that $t \equiv t_1 \circ t_2 \circ t_3$. Also consider choice from elementary tree t' , where t' has $C(t_1)$, $C(t_2)$ and $C(t_3)$ appearing at terminal nodes. We present them in following figures:

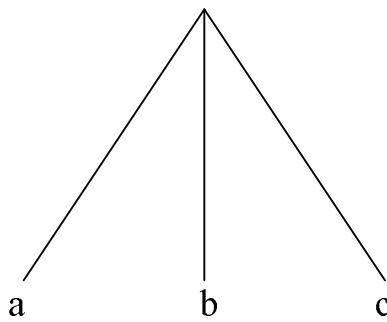


Figure 10: t_1

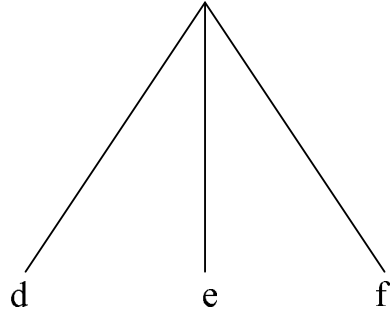


Figure 11: t_2

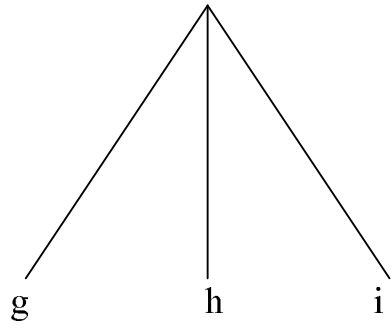


Figure 12: t_3

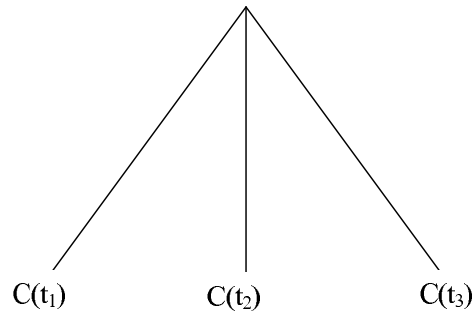


Figure 13: t'

Suppose choice function C satisfies CI. **This implies** $C(t) = C(t')$.

Axiom 2 Repetition Indifference (RI): A Choice Function C satisfies Repetition Indifference (RI) if following holds: if for an arbitrary tree $t \equiv (X^1, X^2, \dots, X^M)$, $C(t) = x_i^j$ then $C(t') = x_i^j$ where t' has been obtained from t by replacing any x_k^l ($x_k^l \neq x_i^j$) by $\beta(x_k^l)$, where x_m^n is the alternative that occupies m -the position from left in the set X^n .

This axiom requires that choice from a tree remains the same, even though some other element has been replaced by an element from its partner set. We provide examples:

Example 2 Consider following binary tree $t \equiv \{(a, b), (c, d)\}$. Suppose that $C(t) = b$. Then RI implies that $C(t') = b$ where $t' \equiv \{(a, b), (c, c)\}$.

Example 3 Consider a ternary tree $t = \{(a, b, c), (d, e, f), (g, h, i)\}$. Suppose that $C(t) = b$. Then $C(t') = b$, where $t' \equiv \{(a, b, a), (d, e, f), (g, h, i)\}$.

Remark 5 RI axiom is motivated by familiar Independence of Irrelevant Alternatives (IIA) axioms used in social choice theory or **Sen's Property α** : for any pair of sets S and T , and for any $x \in S$, if $x \in C(T)$ and $S \subset T$, then $x \in C(S)$. Above axiom requires choice from a tree remains the same, when some other element (i.e. the element which is not chosen from the tree) is replaced by an element from its partner set. Thus according to RI, choice from a larger set $X(t)$ (set of elements appearing at terminal nodes of t) remains the same if available set is contracted to a subset $X(t')$ (set of elements appearing at terminal nodes of t'), such that $C(t) \in X(t')$. Also t' is obtained from t by replacing some element (which is not $C(t)$) by an element from its partner set. This is clearly in the spirit to IIA or Sen's Property α .

We will characterize all choice functions satisfying CI and RI. We first show however that these conditions are independent.

Proposition 1 CI and RI are independent.

Proof: We show this in two parts. First we show there exists a choice function from binary trees that satisfies CI but violates RI. Consider the set of alternatives $X = \{a, b, c\}$. Consider the choice function C^* as defined below:

(i) $C^*(a, b) = b$; $C^*(b, a) = a$.

(ii) $C^*(b, c) = c; C^*(c, b) = b.$

(iii) $C^*(a, c) = a; C^*(c, a) = c.$

Choice from any arbitrary binary tree t is obtained by applying the choice function C^* recursively to elementary trees which are concatenated sequentially to form the tree t .

Note that choice function C^* satisfies CI by construction. But we show that this rule does not satisfy RI. Consider the tree $t \equiv \{(a, b), (c, c)\}$. Clearly, $C^*(t) = C^*\{t_1 \circ t_2\}$, where $t_1 \equiv \{(a, b)\}; t_2 \equiv \{(c, c)\}$. Applying the definition of C^* sequentially we get that $C^*(t) = c$. We now replace b by a in above tree and get $t' \equiv \{(a, a), (c, c)\}$. And $C^*(t') = a$, again using C^* for each elementary tree. But this contradicts RI.

Now we show that RI does not imply CI. Consider the following choice function \hat{C} : for any arbitrary binary tree $t \equiv \{t_1 \circ t_2 \circ \dots \circ t_k\}$ or $t \equiv (X^1, \dots, X^M)$, $C(t) = x_j^i$, where x_j^i : j -th alternative from the left in X^i . We claim that this rule satisfies RI. Consider the tree $t : \{(a, b), (c, d)\}$. Clearly $C(t) = b$. Therefore if any element (other than b) is replaced by its partner element, b will still be the outcome in the choice from new tree. But we observe $\hat{C}\{\hat{C}(a, b), \hat{C}(c, d)\} = \hat{C}(b, d) = d$. But $\hat{C}\{(a, b), (c, d)\} = b$. So this rule violates CI. ■

In the following section we characterize choice functions satisfying the axioms introduced in the previous section.

4 CHOICE FROM BINARY TREES

Consider a choice function C^* from elementary binary trees. We can define a binary relation \succeq over the set of alternatives X in the following manner:

for any $a, b \in X$, if

$C^*(a, b) = a, C^*(b, a) = a$, then $a \succ b$;

$C^*(a, b) = a, C^*(b, a) = b$, then $a \sim_1 b$;

$C^*(a, b) = b, C^*(b, a) = a$, then $a \sim_2 b$.

We can define a binary relation \succeq as follows: $x \succeq y$ if $x \succ y$ or $x \sim_i y$, $i = 1, 2$ for any two alternatives $x, y \in X$. This binary relation is complete over the set of alternatives.

On the other hand, every binary relation over X can be extended in an obvious way to a choice function over all binary trees by concatenation and CI axiom.

Our goal in this section is to show that choice functions from binary trees that satisfy the RI and CI axioms have simple structure. Our main result is the following:

Theorem 1 *A choice function from binary trees C satisfies CI and RI if and only if it is Mixed.*

Proof:

Necessity: Let C be a choice function from binary trees and C satisfies CI and RI. Define following binary relations for all $a, b \in X$: (i) $a \sim_1 b$ if and only if $C(a, b) = a, C(b, a) = b$; (ii) $a \sim_2 b$ if and only if $C(a, b) = b, C(b, a) = a$; and (iii) $a \succ b$ if and only if $C(a, b) = C(b, a) = a$. We claim that

- (a) \sim_i is symmetric for $i = 1, 2$;
- (b) \succ is asymmetric;
- (c) \sim_i is transitive for $i = 1, 2$..
- (d) \succ is transitive.

Let us prove above claims. (a) and (b) follow obviously from the definitions of \sim_i and \succ . Let us prove (c). Let $a \sim_1 b, b \sim_1 c$. Consider the following tree $t \equiv \{(a, a), (b, c)\}$. Hence, using the CI axiom $C(t) = C\{C(a, a), C(b, c)\}$. Now from the relations $a \sim_1 b, b \sim_1 c$ we get that $C(t) = C(a, b) = a$. Since C satisfies RI, $C\{(a, a)(c, c)\} = a$. Again applying CI we get $C(a, c) = a$ [Observation (1)]

Now consider the tree $t' \equiv \{(c, c), (b, a)\}$. Using the definition of $\sim_i, i = 1, 2$ and applying CI we get $C(t') = C\{C(c, c), C(b, a)\} = C(c, b) = c$. Now applying RI, replacing b by a we get that $C\{(c, c), (a, a)\} = c$. Therefore CI implies $C(c, a) = c$ [Observation (2)]

From observations (1) and (2) we get that \sim_1 is transitive. We can show similarly that \sim_2 is also transitive. Now we show that \succ is also transitive. Let $a \succ b, b \succ c$. Consider the tree $t \equiv \{(a, a), (b, c)\}$. Applying CI and the definition of \succ we get $C(t) = C\{C(a, a), C(b, c)\} = C(a, b) = a$. Applying RI we get $C\{(a, a), (c, c)\} = a$. Therefore applying CI we get $C(a, c) = a$. On the other hand, consider the tree $t'' \equiv \{(c, b), (a, a)\}$. Applying CI and the definition of \succ we get $C(t'') = C\{C(c, b), C(a, a)\} = C(b, a) = a$. Then RI implies that $C\{(c, c), (a, a)\} = a$. And CI then implies $C(c, a) = a$. Then $a \succ c$. Therefore \succ is transitive.

We now show following holds good for arbitrary elements $\{a, b, c\} \in X$:

CLAIM (1). $a \succ b, b \sim_1 c \implies a \succ c$,

CLAIM (2). $a \succ b, b \sim_2 c \implies a \succ c$,

CLAIM (3). $a \sim_1 b, b \succ c \implies a \succ c$,

CLAIM (4). $a \sim_2 b, b \succ c \implies a \succ c$.

PROOF OF CLAIM (1). Suppose $a \succ b$ and $b \sim_1 c$. Consider following tree $t \equiv \{(a, a), (b, c)\}$. Applying CI and using above binary relations we get $C\{C(a, a), C(b, c)\} = C(a, b) = a$. Applying RI we get $C\{(a, a), (c, c)\} = a$. Thus $C(a, c) = a$. Again considering the tree $t' \equiv \{(b, c)(a, a)\}$ and applying CI we get $C(t') = C(b, a) = a$. Now again applying RI we get $C\{(b, c), (a, a)\} = C\{(c, c), (a, a)\} = a$. So $C(c, a) = a$. Therefore $a \succ c$. We can prove similarly (2).

PROOF OF CLAIM (3). Suppose $a \sim_1 b$, and $b \succ c$. Consider following tree $t \equiv \{(a, b), (b, c)\}$. Applying CI and using definitions of binary relations \sim_1 and \succ we get $C\{C(a, b), C(b, c)\} = C(a, b) = a$. Now applying RI we get $C\{C(a, a), C(c, c)\} = a$. But applying CI $C\{C(a, a), C(c, c)\} = C(a, c)$ and hence $C(a, c) = a$. Therefore for a, c , only possible relations are $a \sim_1 c$ and $a \succ c$. But if $a \sim_1 c$ holds then using symmetry of \sim_1 we get $a \sim_1 c$ implies $c \sim_1 a$. Also using transitivity of \sim_1 and given $a \sim_1 b$ we get $c \sim_1 b$. But this is in contradiction with $b \succ c$. Thus we must have $a \succ c$. We can show similarly that (iv) holds good.

CLAIM (5). it is not possible to have $a \sim_1 b$ and $b \sim_2 c$.

PROOF OF CLAIM (5). Suppose the claim is not true. Consider the tree $t \equiv \{(b, a), (c, c)\}$. Applying CI we get $C(t) = C\{C(b, a), C(c, c)\}$. Using the relation between a, b, c we get $C(t) = C(b, c) = c$. Now applying RI, $C\{(a, a), (c, c)\} = c$. Again CI implies that $C(a, c) = c$. We can show $a \sim_2 c$ is not possible. This is because had that been true, then $a \sim_2 c, b \sim_2 c$ or $c \sim_2 b$ (by symmetry of \sim_2) would imply $a \sim_2 b$ by transitivity of \sim_2 . But this contradicts $a \sim_1 b$.

Note that $C(a, c) = c$ rules out the case of $a \succ c$ and $a \sim_1 c$ directly. The only remaining possibility is $c \succ a$. Suppose this is true. Consider following tree $t' \equiv \{(c, c), (a, b)\}$. Applying CI we get that $C(t') = C(c, a) = c$. By applying RI and CI we thus get $C\{(c, c), (a, b)\} = c = C\{(c, c), (b, b)\} = C(c, b)$. But this contradicts $b \sim_2 c$. Therefore we conclude that $a \sim_1 b$ and $b \sim_2 c$ is not possible.

Now define an order \succeq over every pair $\{a, b\} \in X$ as follows: $a \succeq b$ if and only if $a \succ b$

or $a \sim_1 b$ or $a \sim_2 b$. If $a \succeq b$ and $b \succeq$ (i.e. $a \sim b$), then either $a \sim_1 b$ or $a \sim_2 b$. It follows from above claims that \succeq is complete, reflexive and transitive. Thus \succeq is a weak preference ordering over the set of alternatives X . We define an indifference set I as a subset of X such that for any $x, y \in I$ we have $x \succeq y$ and $y \succeq x$. By claim (5), all members in any indifference set are related to each other by binary relation \sim_1 or by \sim_2 , but both cannot hold good. We define an indicator function $\delta(\cdot)$ over indifference sets in following manner: for an indifference set I_i , $\delta(I_i) = 1$ if for any $x, y \in I_i, x \sim_1 y$ and $\delta(I_i) = 2$ if for any $x, y \in I_i, x \sim_2 y$.

Consider a tree $t \in \Gamma^2$ and suppose $C(t) = x_i^j$. Thus the i -th alternative from the left (since t is binary, $i = 1$ or $i = 2$) in the j -th set from the left in tree t , i.e. X^j , is chosen. We claim that $x_i^j \in M(X(t), \succeq)$. Let us prove it by contradiction. Suppose $x_i^j \notin M(X(t), \succeq)$. Therefore $\exists x_m^n \in X(t)$ such that $x_m^n \succ x_i^j$. We now define an algorithm (Concatenation Algorithm): we apply CI sequentially on trees which are recursively concatenated to form t . Since t is binary tree, X^n has two alternatives, one of which is x_m^n . Suppose the other alternative of X^n is x_l^n ($l, m = 1, 2; l \neq m$). Consider the choice from the elementary binary tree represented by X^n . If x_m^n is chosen then $x_m^n \succeq x_l^n$, or if x_l^n is chosen then $x_l^n \succeq x_m^n$. We denote the chosen alternative from this first round by $x(1)$. In the next round of applying CI, we observe the choice problem from an elementary tree comprising of two alternatives one of which must be $x(1)$ ($= x_m^n$ or x_l^n , depending on the outcome in the last round of choice problem). The outcome from present round of choice problem, say $x(2)$, will certainly satisfy following: $x(2) \succeq x(1)$, which implies $x(2) \succeq x_m^n$ or $x' \succeq x_l^n$, depending on the outcome in the last round of sequential choice problem. We observe, anyways $x(2) \succeq x_m^n$ by transitivity property of \succeq . In fact this holds for outcome for any p -th round $x(p)$, i.e. $x(p) \succeq x_m^n$. Thus we carry on applying CI sequentially to reach the last round, say r -th round, where we face the problem $C\{C(t_1), C(t_2)\}$ such that $t \equiv (t_1 t_2)$ and $x(r-1)$ is the alternative that entered this last round of exercise we are carrying recursively and we have $x(r)$ that is the outcome of $C\{C(t_1), C(t_2)\}$. Thus we have $x(r) \succeq x_m^n$. But since $C(t) = x_i^j$, we have $x(r) = x_i^j$. Thus $x_i^j \succeq x_m^n$. This is in contradiction to what we assumed, i.e. $x_m^n \succ x_i^j$.

Now let us characterize $C(t)$ when $M(X(t), \succeq)$ is not singleton. We have $C(t) = x_i^j$ and we have proved that $x_i^j \in M(X(t), \succeq)$. Consider any other alternative from the set $M(X(t), \succeq)$. Suppose $x_{i'}^j \in M(X(t), \succeq)$. We get from the definition of a maximal set $x_i^j \succeq x_{i'}^j$ and $x_{i'}^j \succeq x_i^j$. Thus maximal set is an indifference set characterized by either \sim_1 or \sim_2 . W.l.o.g. we assume $x_i^j \sim_1 x_{i'}^j$. Thus $\delta(M(X(t), \succeq)) = 1$. We claim that the left-most alternative in $M(X(t), \succeq)$ is chosen. We prove this as follows:

Suppose the left-most alternative in $M(X(t), \succeq)$ is x_i^j , i.e. i -th alternative from the left in the set X^j . We note that for any alternative x_m^n which is to the left of x_i^j , we have $x_i^j \succ x_m^n$, since x_i^j is the left-most alternative of $M(X(t), \succeq)$ in the tree t . We now consider a concatenation algorithm as described earlier starting from x_i^j . We note that for each round

p , $x(p)$ (the alternative that is chosen from the choice problem in p -th round) is x_i^j only. This is because in each round p , starting from $p = 1$, if the elementary binary tree is of the form $\{x_i^j, y\}$, then choice is x_i^j . This is because if $y \in M(X(t), \succeq)$, then it must be that $x_i^j \sim_1 y$ and thus x_i^j is chosen. If $y \notin M(X(t), \succeq)$, then $x_i^j \succ y$ and thus x_i^j is chosen. If the elementary binary tree in round p is of the form $\{y, x_i^j\}$, then choice is again x_i^j . This is because if $y \notin M(X(t), \succeq)$ then $x_i^j \succ y$ and thus x_i^j is chosen. Now we claim that the tree cannot be of the form $\{y, x_i^j\}$ if $x_i^j \sim_1 y$. This is ensured from the fact that in t , x_i^j is the left-most alternative in the set $M(X(t), \succeq)$ and from the structure of concatenation algorithm. We note that following this algorithm in the last round we get $x(r)$ which is $C(t)$. Thus $C(t) = x_i^j$ only.

Thus we have shown that if choice function C from binary trees satisfies CI and RI then there exists a weak order such that for any tree $t \in \Gamma^2$, we have $C(t) \in M(X(t), \succeq)$. If $M(X(t), \succeq)$ contains more than one element, then the left-most alternative in $M(X(t), \succeq)$ is chosen if $\delta(x) = 1$, for $x \in M(X(t), \succeq)$ or the right-most alternative in $M(X(t), \succeq)$ is chosen if $\delta(x) = 2$, for $x \in M(X(t), \succeq)$. We then observe that C is **Mixed**.

Sufficiency: If choice function from binary trees C is characterized by such an weak order over X , so that chosen alternative from any tree $t \in \Gamma^2$ is the maximal alternative from $X(t)$ according to the weak order or the extreme left or right one (according to an indicator function) from the maximal set, then it is easy to verify that C satisfies CI. Using an algorithm similar to concatenation algorithm described above one can show choice made from a tree $t \equiv \{t_1 ot_2\}$ coincides with choice made from a tree t' , formed with alternatives $C(t_1)$ and $C(t_2)$ as follows: $t' \equiv (C(t_1), C(t_2))$. This is because $M(X(t), \succeq)$ is same as $M(X(t'), \succeq)$. If maximal set is not singleton, the tie-breaking rule through indicator function is also same. It is easy to check C satisfies RI: suppose for $t \in \Gamma^2$, $C(t) = x$. Thus $x \in M(X(t), \succeq)$ or if $M(X(t), \succeq)$ is not singleton, then tie is broken according to the indicator function $\delta(M(X(t), \succeq))$. W.l.o.g. assume left-most alternative is chosen from $M(X(t), \succeq)$. Suppose any alternative $x' (\neq x)$ is replaced by its partner element and the new tree be denoted by t' . We note that $M(X(t'), \succeq)$ is still x if $M(X(t), \succeq) = x$. If $M(X(t), \succeq)$ is not singleton (and x is the left-most or the right-most alternative in $M(X(t), \succeq)$ according to the indicator function), then we observe $M(X(t'), \succeq) \subset M(X(t), \succeq)$. Thus the indicator function chooses the same alternative, i.e. x , because even after above replacement x remains the left-most alternative in $M(X(t), \succeq)$. Thus we show C satisfies RI. ■

5 CHOICE FROM k -ARY TREES

Suppose $\Gamma^k(B)$ denotes the set of all k -ary trees formed with all alternatives from the set B and $\Gamma_e^k(B)$ denotes the set of all elementary k -ary trees formed with all alternatives from the set B . Here $B \subset X$. We note that we have already denoted the set of all alternatives which appear at the terminal nodes of any tree t by $X(t)$. Therefore for any tree $t \in \Gamma^k(B)$, $X(t) = B$.

We first consider choice functions from k -ary trees formed with only two alternatives, i.e. trees with $|X(t)| = 2$ for any tree t . According to the notations set earlier in this section, $\Gamma^k(\{a, b\})$ denotes set of all k -ary trees formed with any two alternatives $\{a, b\} \in A$ and $\Gamma_e^k(\{a, b\})$ denotes set of all elementary k -ary trees formed with $\{a, b\} \in A$. Consider choices made from these trees by different choice functions. We can define binary relations between a and b :

Suppose C is any choice function from elementary k -ary trees.

Definition 20 $a \succ b$ if and only if $C(t) = a$ for any $t \in \Gamma_e^k(\{a, b\})$.

Thus binary relation \succ is induced by above choice function C . On the other hand we also say that above binary relation \succ induces choice function C as defined above.

Definition 21 $a \sim_r b$ if and only if $C(t) = x(t, r)$, $1 \leq r \leq k$ for any $t \in \Gamma_e^k(\{a, b\})$ [where $x(t, r)$ is the alternative placed in r -th position from the left in the elementary tree t].

Thus binary relation \sim_r is induced by above choice function C . Also binary relation \sim_r induces choice function C .

Definition 22 $a \leftrightarrow b$ if and only if following holds: (i) $\exists t_1, t_2 \in \Gamma_e^k(\{a, b\})$ such that $C(t_1) = a, C(t_2) = b$; (ii) For any $r, 1 \leq r \leq k$, $\exists t_1 \in \Gamma_e^k(\{a, b\})$ such that $C(t_1) \neq x(t_1, r)$; and (iii) Suppose for any arbitrary $t_1 \in \Gamma_e^k(\{a, b\})$ we have $C(t_1) = a$ (w.l.o.g.) where a occupies at least one position in the tree t_1 . Suppose w.l.o.g. that position is r th position from the left, i.e. $x(t_1, r) = a$. Consider any other tree $t_2 \in \Gamma_e^k(\{a, b\})$ such that $x(t_2, r) = a$ and $C(t_2) = b$. Then there must exist at least one position $l; 1 \leq l \leq k, l \neq r$ such that $x(t_1, l) = a$ and $x(t_2, l) = b$.

Thus binary relation \leftrightarrow is induced by above choice function C and vice-versa.

Now we can show all above choice functions induced by binary relations \succ , \sim_r and \leftrightarrow satisfy RI. Moreover we can show that these choice functions are the *only* choice functions from elementary k -ary trees that satisfy RI. We state and prove this in following lemma:

Lemma 1 *Suppose C is a choice function from trees in $\Gamma_e^k(\{a, b\})$. C satisfies RI if and only if C induces either of the following binary relations: (i) \succ , (ii) \sim_r ; $1 \leq r \leq k$ or (iii) \leftrightarrow .*

Proof:

Sufficiency part is easy to check. We here consider the necessity part. Suppose C is any choice function from trees in $\Gamma_e^k(\{a, b\})$ and satisfies RI. We consider all possible cases of the outcome sets produced by C from trees in $\Gamma_e^k(\{a, b\})$:

Case (i) One possibility is that $C(t) = a$ (or b) for any $t \in \Gamma_e^k(\{a, b\})$. We can show that C satisfies RI. To prove this by contradiction we initially assume that C does not satisfy RI. But for any $t \in \Gamma_e^k(\{a, b\})$ we have $C(t) = a$. Consider any tree $t_1 \in \Gamma_e^k(\{a, b\})$. If we replace b in t_1 by a [according to RI we need to replace any alternative in t_1 which is not the outcome (i.e. b) by its *Partner element* (i.e. a)] let the resulting tree be t_2 . But $C(t_2) = a$ again. Thus C satisfies RI.

We note that case (i) implies that $a \succ b$ (or $b \succ a$ if $C(t) = b$ for any $t \in \Gamma_e^k(\{a, b\})$). Thus under case (i) C induces binary relation \succ as it follows from Definition 1.

Case (ii) For any choice function C not falling under case (i), there exist two mutually exclusive sets of trees $\Gamma_{e_1}^k(\{a, b\})$ and $\Gamma_{e_2}^k(\{a, b\})$ from $\Gamma_e^k(\{a, b\})$, such that $\Gamma_e^k(\{a, b\}) = \Gamma_{e_1}^k(\{a, b\}) \cup \Gamma_{e_2}^k(\{a, b\})$ and $C(t) = a, \forall t \in \Gamma_{e_1}^k(\{a, b\})$ and $C(t) = b, \forall t \in \Gamma_{e_2}^k(\{a, b\})$. Since C satisfies RI we can show that it satisfies following property:

Consider any $t_1 \in \Gamma_{e_1}^k(\{a, b\})$ and $t_2 \in \Gamma_{e_2}^k(\{a, b\})$. Suppose $\exists r, 1 \leq r \leq k$ such that $x(t_1, r) = x(t_2, r)$. Without loss of generality assume that $x(t_1, r) = x(t_2, r) = a$. Then there must exist at least one position (from the left) $l; 1 \leq l \leq k, l \neq r$ such that $x(t_1, l) = a$ and $x(t_2, l) = b$. Let us call this Property $-\alpha$.

We show it by contradiction. Suppose $t_1 \in \Gamma_{e_1}^k(\{a, b\})$ and $t_2 \in \Gamma_{e_2}^k(\{a, b\})$ are two k -ary trees such that $x(t_1, r) = x(t_2, r) = a$ for some $r, 1 \leq r \leq k$. Also suppose that there is no $l(1 \leq l \leq k, l \neq r)$ such that $x(t_1, l) = a$ and $x(t_2, l) = b$. Assume that the set of all positions (from the left) occupied by b in t_2 be P . It follows from above that in t_1 these positions of the set P are also occupied by b . We replace all elements which are placed in positions other than those in P by a in t_1 and let the new tree be t_3 . By RI we get that $C(t_3) = a$. But we

can check that t_3 is actually t_2 . But we have assumed initially that $C(t_2) = b$. This leads to a contradiction.

We can further categorize the choice functions under case (ii) as follows:

(a) Given choice function C , $\exists i, 1 \leq i \leq k$ such that $x(t, i) = a, \forall t \in \Gamma_{e_1}^k(\{a, b\})$ and $x(t, i) = b, \forall t \in \Gamma_{e_2}^k(\{a, b\})$. Thus $C(t) = x(t, i) \forall t \in \Gamma_e^k(\{a, b\})$. We can easily verify that C satisfies Property α . This is because C always chooses the alternative which occupies i -th position from the left in any tree. Thus for any two trees $t_1 \in \Gamma_{e_1}^k(\{a, b\})$ and $t_2 \in \Gamma_{e_2}^k(\{a, b\})$ such that $x(t_1, g) = x(t_2, g) (1 \leq g \leq k)$ we have i -th position from the left that is occupied by a in all trees from $\Gamma_{e_1}^k(\{a, b\})$ and occupied by b in all trees from $\Gamma_{e_2}^k(\{a, b\})$. Thus $x(t_1, i) = a$ and $x(t_2, i) = b$ since $t_1 \in \Gamma_{e_1}^k(\{a, b\})$ and $t_2 \in \Gamma_{e_2}^k(\{a, b\})$.

Thus there can be k different choice functions under this category (a), each choice function picking the alternative occupying the i -th position from the left in any tree and $1 \leq i \leq k$.

We note that a choice function C that always chooses the alternative that occupies i -th position in a tree induces binary relation \sim_i . It thus follows from Definition 2 that all choice functions under this category (a) induce binary relations $\{\sim_i\}_1^k$.

(b) For any other choice function C under case (ii) but that does not fall under category (a) following holds true: for any position i in the k -ary tree from the left ($1 \leq i \leq k$), there is a tree, say $t \in \Gamma_e^k(\{a, b\})$, such that $C(t) \neq x(t, i)$ (We note that this is (ii) of definition of binary relation \leftrightarrow). C must also satisfy Property α .

We can also show C induces binary relation \leftrightarrow . This follows from definition of \leftrightarrow (Definition 3). C clearly satisfies (i) of definition 3 since $C(t) \neq a$ (or b) $\forall t \in \Gamma_e^k(\{a, b\})$. Also (ii) of definition 3 is also satisfied as noted above. Also (iii) of definition 3 is satisfied because C satisfies Property α .

■

Above lemma implies that a choice function C from k -ary trees, formed with alternatives from set X and satisfies RI, must induce either of following binary relations between any two alternatives $a, b \in X$: $\succ, sim_r (1 \leq r \leq k)$ or \leftrightarrow . This is because such C incorporates choice from elementary k -ary trees formed with any $a, b \in X$ as well.

We now consider any choice function C from all k -ary trees formed with elements from X and also suppose C satisfies RI and CI. From above lemma we note that C induces either of following binary relations over the set X : $\succ, \sim_r (1 \leq r \leq k)$ and \leftrightarrow . Before we further characterize binary relations \succ, \sim_i and \leftrightarrow over set X we provide a definition and prove a

lemma.

Given definitions of binary relations \succ , \sim_r and \leftrightarrow over set X , we define a binary relation R as follows:

Definition 23 A binary relation R over the set X , where $|X| > 2$, is called **Single Maximal Element Preference** with respect to k -ary trees, if following holds good:

- For any $a, b \in X$, aRb if and only if $a \succ b$ or $a \sim_r b$ ($1 \leq r \leq k$) or $a \leftrightarrow b$.
- There exists a subset $X' \subset X$ such that $\forall x_r, x_s \in X'$, we have $x_r R x_s \implies x_r \sim_r x_s$ ($1 \leq r \leq k$) or $x_r \leftrightarrow x_s$. If $x_r \leftrightarrow x_s$, then $X' = \{x_r, x_s\}$, i.e. cardinality of the set X' cannot be more than 2. For all $x_m, x_n \in X - X'$ we have either $x_m \succ x_n$ or $x_n \succ x_m$ and for any $x_i \in (X - X')$, $x_j \in X'$ we have $x_i \succ x_j$. Thus $\forall x_r, x_s \in X'$ it must be that $x_r R x_s$ and $x_s R x_r$. Also for any $x_m, x_n \in X - X'$ if $x_m R x_n$ then it must not be $x_n R x_m$. Also for any $x_i \in (X - X')$, $x_j \in X'$ we have $x_i R x_j$ but $x_j R x_i$ does not hold. Thus R is anti-symmetric over $X - X'$ but symmetric over X' .
- R is transitive, i.e. for any $x_1, x_2, x_3 \in (X - X')$, such that $x_1 \succ x_2$ and $x_2 \succ x_3$ we have $x_1 \succ x_3$.

The domain containing such preferences is called **Single Maximal Element Domain**.

Remark 6 We observe that for $|X| = 2$ definition of single maximal element preference implies R is pure preference \succ .

Definition 24 A choice function from k -ary trees is **Single Maximal Element Preference based (SMEP)** if there exists a Single Maximal Element Preference R (w.r.t. k -ary trees) on X such that for all trees $t \in \Gamma^k(B)$,

$$C(t) = (M(X(t), R)), \text{ where } X(t) = B.$$

Suppose $(M(X(t), R))$ is not singleton, i.e. either (i) for any $x_1, x_2 \in (M(X(t), R))$, $x_1 \sim_r x_2$ where $1 \leq r \leq k$; or (ii) for $x_1, x_2 \in (M(X(t), R))$, $x_1 \leftrightarrow x_2$ and $|(M(X(t), R))| = 2$. If (i) holds then $C(t) = x(r; t)$ $1 \leq r \leq k$, i.e. the element in $X(t)$ that can be reached in the tree t by picking the r -th branch at every non-terminal node, starting from the initial node, is chosen. In case (ii) holds, $C(t) = x$, where x is obtained as follows: apply choice function induced by \leftrightarrow (let us denote it by C_{\leftrightarrow}) to all elementary trees which are horizontally concatenated to form some k -ary trees and which are again concatenated to form some k -ary

trees and continues like this to finally form the tree t . Recursive use of choice function C_{\rightarrow} to all stages leads to choice from final elementary k -ary tree and x results from applying C_{\rightarrow} to this tree.

Remark 7 It follows from the last remark that SMEP choice functions are defined only when $|X| > 2$.

Example 4 Suppose C is a SMEP from ternary trees, which is characterized by Single Maximal Element Preference R . Consider set of alternatives $X = \{a, b, c, d, e, f\}$. Suppose the Single Maximal Element Preference R over X is such that, R is anti-symmetric over $\{a, b, c\}$ and symmetric over $\{d, e, f\}$. Also for each element in $\{a, b, c\}$, say a , and for each element in $\{d, e, f\}$, say d , we have aRd but not dRa . Also suppose for any two elements in $\{d, e, f\}$, say d and e , R is characterized by $d \sim_2 e$. Consider following trees:

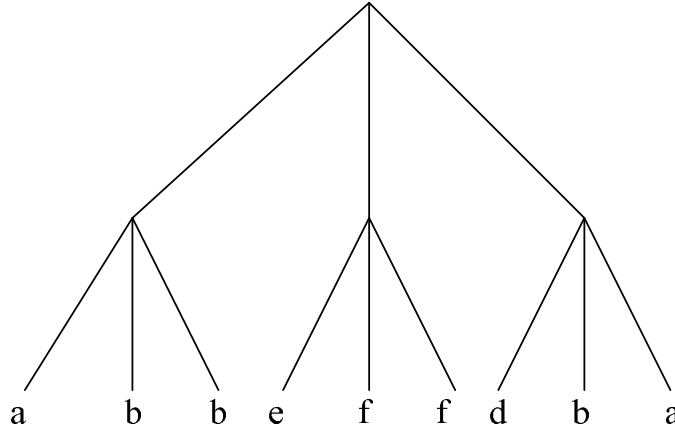


Figure 14: Tree: t

We observe that $X(t) = \{a, b, d, e, f\}$ and $M(X(t), R) = a$. Thus $C(t) = a$.

For t' , we have $X(t') = \{d, e, f\}$. Here $M(X(t'), R) = \{d, e, f\}$. Elements in this set are related to each other by \sim_2 . Thus $C(t')$ is the alternative which can be reached by choosing the second branch at every non-terminal node, starting from the initial node. Thus $C(t') = d$.

Example 5 Suppose C is a SMEP from ternary trees, which is characterized by Single Maximal Element Preference R' . Consider set of alternatives $X = \{a, b, c, d, e, f\}$. Suppose the Single Maximal Element Preference R' over X is such that, R' is anti-symmetric over $\{a, b, c, d\}$ and symmetric over $\{e, f\}$. Also for each element in $\{a, b, c, d\}$, say a , and for each element in $\{e, f\}$, say e , we have $aR'e$ but not $eR'a$. Also suppose we have $e \leftrightarrow f$ such that we have following:

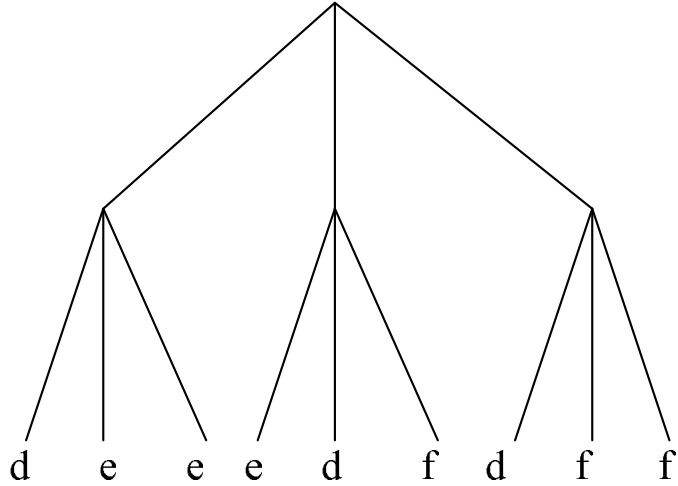


Figure 15: Tree: t'

$$C(e, f, f) = e, C(f, e, e) = e.$$

$$C(e, e, f) = e, C(f, f, e) = f.$$

$$C(e, f, e) = e, C(f, e, f) = f.$$

Consider choice from tree t'' in following figure.

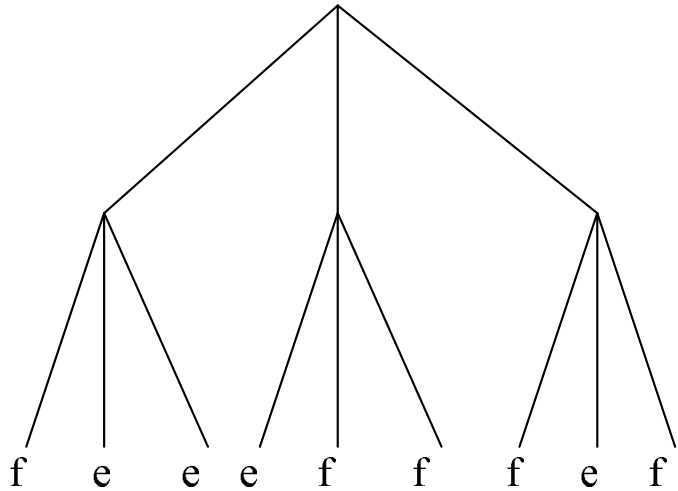


Figure 16: Tree: t''

We observe here $C(t'') = e$.

We now state and prove an important lemma for choice from k -ary trees:

Lemma 2 Consider trees in Γ^k and choice function C from trees in Γ^k that satisfies CI

and RI. Suppose for some $t_1 \in \Gamma_e^k(X')$ ($X' \subset X$), $C(t_1) = x_1$ where $x_1 \in X(t_1)$. Without loss of generality suppose x_1 occupies the i -th position from the left in t_1 , i.e. $x_i(t_1) = x_1$. Consider $t_2 \in \Gamma_e^k(X'')(X'' \subset X)$ such that $C(t_2) = x_3$ and $x_i(t_2) \neq x_3$. Also suppose $x_1, x_3 \in X(t_1) \cap X(t_2)$. Then there must exist at least one position (from the left in a k -ary tree), say j -th position ($1 \leq j \leq k$) such that $x_j(t_1) = x_1$ and $x_j(t_2) = x_3$.

Proof: We prove by contradiction. Suppose $C(t_1) = x_1$, $C(t_2) = x_3$ and for each position l (from the left in a k -ary tree) such that $x_l(t_2) = x_3$ we have $x_l(t_1) \neq x_1$. Also we have $x_1, x_3 \in X(t_1) \cap X(t_2)$. We call the set of all such positions I . Suppose we replace all elements in t_1 that occupy positions of set I by x_3 (We note that all these places are occupied by other elements apart from x_1 in t_1) and other positions which are not occupied by x_1 in t_1 by x_1 . Suppose we obtain tree t_3 . Thus applying RI we get that $C(t_3) = x_1$. Also we replace i -th position in t_2 by x_1 and all other positions apart from positions in set I by x_1 . Again we obtain t_3 by this process but applying RI sequentially we get $C(t_3) = x_3$. This is contradictory to above $C(t_3) = x_1$. We observe this generalizes α - Property discussed in lemma 1. \blacksquare

In following lemmas we characterize binary relations \succ, \sim_r ($1 \leq r \leq k$) and \leftrightarrow over the set X when choice function C from k -ary trees in Γ^k satisfies CI and RI.

Lemma 3 \succ exhibits transitive property, i.e. for any $a, b, c \in X$, $a \succ b$ and $b \succ c$ implies $a \succ c$.

Proof: Suppose $a \succ b$ and $b \succ c$. Consider trees in $\Gamma^k(\{a, b, c\})$. In particular consider a tree $t' \in \Gamma^k(\{a, b, c\})$ such that $t' \equiv (X^1, X^2, \dots, X^k)$, where each X^i represents an elementary k -ary tree. Suppose for any arbitrary i ($1 \leq i \leq k$), X^i represents an elementary k -ary tree, in which a occupies only one arbitrary position and other positions are occupied by b . Therefore suppose $x_r^i = a$, $x_s^i = b$; ($1 \leq r, s \leq k$, $r \neq s$). Also suppose X^j (for any j : $1 \leq j \leq k$, $i \neq j$) is any arbitrary elementary k -ary tree consisting of only b and c .

Consider $C(t)$. Given C satisfies CI and since $a \succ b$ and $b \succ c$, we have $C(t) = C(t')$. Here t' is an elementary k -ary tree, such that $x(t', i) = a$ and $x(t', l) = b$ for any l : $1 \leq l \leq k$, $l \neq i$. Since $a \succ b$, we have $C(t') = C(t) = a$.

We now replace b by c in each X^j (for any j : $1 \leq j \leq k$, $i \neq j$) of tree $t' \equiv (X^1, X^2, \dots, X^k)$ and get tree t'' . Applying RI we get $C(t'') = a$. Again applying CI on $C(t'')$, and using $a \succ b$ and $b \succ c$, we get $C(t'') = C(t''') = a$. Here t''' is an elementary k -ary tree with $x(t''', i) = a$ and $x(t''', l) = c$ for any l : $1 \leq l \leq k$, $l \neq i$. We observe in t''' , only one position (i -th position from the left) is occupied by a and this position is arbitrary. Thus we conclude $a \succ c$.

■

Lemma 4 \sim_r exhibits transitive property, i.e. for any $a, b, c \in X$, $a \sim_r b$ and $b \sim_r c$ implies $a \sim_r c$.

Proof: Suppose $a \sim_r b$ and $b \sim_r c$. Consider trees in $\Gamma^k(\{a, b, c\})$. In particular consider a tree $t' \in \Gamma^k(\{a, b, c\})$ such that $t' \equiv (X^1, X^2, \dots, X^k)$, where each X^i represents an elementary k -ary tree. Suppose X^r is such that, $x_r^r = a$ and $x_l^r = b$ for any $l: 1 \leq l \leq k, l \neq r$. Also suppose for any X^j ($j: 1 \leq j \leq k, j \neq r$) we have $x_r^j = b$ and $x_l^j = c$ for any $l: 1 \leq l \leq k, l \neq r$.

Consider $C(t)$. Given C satisfies CI and since $a \sim_r b$ and $b \sim_r c$, we have $C(t) = C(t')$. Here t' is an elementary k -ary tree, such that $x(t', r) = a$ and $x(t', l) = b$ for any $l: 1 \leq l \leq k, l \neq r$. Since $a \sim_r b$, we have $C(t') = C(t) = a$.

We now replace b by c in each X^j (for any $j: 1 \leq j \leq k, r \neq j$) of tree $t \equiv (X^1, X^2, \dots, X^k)$ and get tree t'' . Applying RI we get $C(t'') = a$. Again applying CI on $C(t'')$, and using $a \sim_r b$, we get $C(t'') = C(t''') = a$. Here t''' is an elementary k -ary tree with $x(t''', r) = a$ and $x(t''', l) = c$ for any $l: 1 \leq l \leq k, l \neq r$. We observe in t''' , r -th position from the left is occupied by a . Similarly we can show $C(t''') = c$, where $x(t''', r) = c$ and $x(t''', l) = a$ for any $l: 1 \leq l \leq k, l \neq r$. Therefore we prove $a \sim_r c$.

■

Lemma 5 For any $a, b, c \in X$ if $a \sim_i b$ ($1 \leq i \leq k$) then it is not possible to have $b \sim_j c$ ($1 \leq j \leq k, i \neq j$).

Proof: Suppose $a \sim_i b$ and $b \sim_j c$ ($1 \leq i \leq k; 1 \leq j \leq k$ and $i \neq j$). Consider a tree t such that $x(t, i) = c, x(t, j) = a$ and $x(t, l) = b$, for any $1 \leq l \leq k, l \neq i, j$. Consider $C(t)$. We observe $C(t) \neq a$. Because if $C(t) = a$, then by RI, we get $C(t') = a$ by replacing c by b , where $x(t', i) = b, x(t', j) = a$ and $x(t', l) = b$, for any $l, 1 \leq l \leq k, l \neq i, j$. But given $a \sim_i b$, we have $C(t') = b$, because $x(t', i) = b$.

Also $C(t) \neq c$ by similar argument: if $C(t) = c$, then we use RI and replace a by b in t to get t'' such that $C(t'') = c$. But $x(t'', j) = b$ and $b \sim_j c$, implying $C(t'') = b$. Thus we have a contradiction. Similar argument shows $C(t) \neq b$ also. Thus it is impossible to have $a \sim_i b$ and $b \sim_j c$ ($i \neq j$).

■

Lemma 6 For any $a, b, c \in X$ if $a \sim_r b$ ($1 \leq r \leq k$) then it is not possible to have $a \succ c$.

Proof: On the contrary suppose for some $a, b, c \in X$ we have $a \sim_r b$ and $a \succ c$. Consider choice from an elementary tree t such that $x(t, r) = c$, $x(t, m) = a$ ($m \neq r$) and $x(t, n) = b$ for any n , $n \neq r, m$; $1 \leq n \leq k$. We consider possibilities for $C(t)$.

Clearly $C(t) \neq c$. This is because if $C(t) = c$, then we can use RI repetitively and replace b by a in all positions $n \neq r, m$; $1 \leq n \leq k$ and we get $C(t') = c$, where t' is the tree we get by replacing all b by a in above manner. But in t' all positions are occupied by a except r -th position which is occupied by c and we have $a \succ c$. Thus $C(t')$ should be a by the definition of \succ . Thus there is a contradiction.

Also if $C(t) = b$, then we use RI and replace c by a in r -th position to get tree t'' . Thus $x(t'', r) = a$. B RI we get $C(t'') = b$. But we have $a \sim_r b$ and thus following definition of \sim_r we must have $C(t'') = a$. This is a contradiction.

Similarly if we have $C(t) = a$, then we use RI and replace c by b in r -th position to get tree t''' . Thus $x(t''', r) = a$. Applying RI, we get $C(t''') = b$. But we have $a \sim_r b$ and following definition of \sim_r , we have $C(t''') = b$. This is a contradiction.

Thus we can show that for any $a, b, c \in X$ if $a \sim_r b$ ($1 \leq r \leq k$) then it is not possible to have $a \succ c$.

■

Lemma 7 For any $a, b, c \in X$ if $a \leftrightarrow b$ then it is not possible to have $b \succ c$.

Proof: On the contrary suppose for some $a, b, c \in X$ we have $a \leftrightarrow b$ and $b \succ c$. First consider the trees in $\Gamma_e^k\{a, b\}$. We have $a \leftrightarrow b$. Using definition of \leftrightarrow we get the following:

$\exists t_1 \in \Gamma_e^k\{a, b\}$ such that $C(t_1) = a$. This is because if $a \leftrightarrow b$, then it cannot be that $C(t) = a$ (or b) for all $t \in \Gamma_e^k\{a, b\}$. In t_1 there must be at least one position (say r -th position from left) such that $x(t_1, r) = a$. We consider the choices made by C from all trees (except t_1) in $\Gamma_e^k\{a, b\}$ such that r -th position in these trees are occupied by a and also check if the choice from any of these trees is b . There can be two possibilities:

- (i) there exists a tree t_2 such that $x(t_2, r) = a$ and $C(t_2) = b$; Or

(ii) for all such trees in which a occupies r -th position, choice function results in a , i.e. for any $t \in \Gamma_e^k\{a, b\}$ such that $x(t, r) = a$ we have $C(t) = a$.

In case (ii) holds, we understand that choice in all elementary k -ary trees (formed with $\{a, b\}$) in which r -th position is occupied by a , is a . We note that given any position i from the left ($1 \leq i \leq k$), all elementary k -ary trees (formed with $\{a, b\}$) can be classified into two groups: in one group of trees i -th position is occupied by a and in the other group the i -th position is occupied by b . Thus in case (ii) holds we must have a k -ary tree (say t_3) from $\Gamma_e^k\{a, b\}$ such that $x(t_3, r) = b$ and $C(t_3) = a$. This is because if (ii) holds, for any $t \in \Gamma_e^k\{a, b\}$, $x(t, r) = a$ implies $C(t) = a$. Thus for all trees $t \in \Gamma_e^k\{a, b\}$, for which $x(t, r) = b$, if $C(t) = b$ then $C(t) = x(r; t) \forall t \in \Gamma_e^k\{a, b\}$. But this implies $a \sim_r b$. Thus there must be t_3 such that $x(t_3, r) = b$ and $C(t_3) = a$. Obviously since $C(t) \neq a \forall t \in \Gamma_e^k\{a, b\}$, we must have a tree $t_4 \in \Gamma_e^k\{a, b\}$ such that $x(t_4, r) = b$ and $C(t_4) = b$.

Thus given $a \leftrightarrow b$, we must either have (i) $\exists t_1, t_2 \in \Gamma_e^k\{a, b\}$ such that $x(t_1, r) = x(t_2, r) = a$ and $C(t_1) = a, C(t_2) = b$; or (ii) $\exists t_3, t_4 \in \Gamma_e^k\{a, b\}$ such that $x(t_4, r) = x(t_3, r) = b$ and $C(t_4) = b, C(t_3) = a$.

W.l.o.g. we assume $\exists t_1, t_2 \in \Gamma_e^k\{a, b\}$, such that $x(t_1, r) = x(t_2, r) = a$, and $C(t_1) = a, C(t_2) = b$. From definition of \leftrightarrow , we also note there must exist at least one position, say $l; 1 \leq l \leq k, l \neq r$ such that $x(t_1, l) = a$ and $x(t_2, l) = b$. Also suppose I is the set of all such positions in a k -ary tree, such that for each position i , we have $x(t_1, i) = b$. Clearly $r, l \notin I$.

Consider a tree $t^* \in \Gamma^k\{a, b, c\}$ such that $x(t^*, r) = a, x(t^*, l) = c$. Also $x(t^*, i) = b, \forall i \in I$ and $x(t^*, j) = c, 1 \leq j \leq k; j \neq r, l; j \notin I$.

Our claim is $C(t^*) \neq c$. This can be shown using RI and $b \succ c$. We can also show $C(t^*) \neq b$. This is because if $C(t^*) = b$, then we can replace all c in t^* by a to get t^{**} and applying RI repetitively we get $C(t^{**}) = b$. But we note that $t^{**} = t_1$ and $C(t_1) = a$. This leads to a contradiction. Now let us show $C(t^*) \neq a$ by contradiction. Suppose $C(t^*) = a$. We can replace c by b in t^* in all positions occupied by c in t^* to get t^{***} and apply RI to get $C(t^{***}) = a$. We also know $C(t_2) = b$. We replace a by b in t_2 in all positions in t_2 occupied by a , except the r -th position and get t^{****} . Applying RI repetitively we get $C(t^{****}) = b$. But we observe $t^{***} = t^{****}$ and we have $C(t^{***}) = a$. Hence we get a contradiction and prove the lemma. ■

Lemma 8 For any $a, b, c \in X$ if $a \leftrightarrow b$ then it is not possible to have $b \leftrightarrow c$.

Proof: We begin the proof with an observation. Consider elementary k -ary trees formed with $\{a, b, c\}$, i.e. trees in the set $\Gamma^k\{a, b, c\}$. We have $a \leftrightarrow b$ and $b \leftrightarrow c$. Consider any tree $t_1 \in \Gamma^k\{a, b, c\}$, such that an alternative (w.l.o.g. suppose b) occupies only one position (suppose i -th position, $1 \leq i \leq k$) in the tree t_1 and other positions are occupied by $\{a, c\}$. We claim $C(t_1) \neq b$. To prove this, on the contrary, we assume $C(t_1) = b$. Consider $C(t_2)$ where $x(t_2, i) = b$, $x(t_2, j) = c$ and $x(t_2, l) = a$; $1 \leq l \leq k$; $l \neq i, j$. Since $C(t_1) = b$ and in t_1 , i -th position is occupied by b , we can use RI to get $C(t_2) = b$ also. Now consider $C(t_3)$ such that $x(t_3, i) = c$, $x(t_3, j) = b$ and $x(t_3, l) = a$; $1 \leq l \leq k$; $l \neq i, j$. We consider possibilities for $C(t_3)$.

Assume $C(t_3) = c$. Then consider $C(t_4)$ such that $x(t_4, i) = c$, $x(t_4, j) = b$; $1 \leq j \leq k$; $i \neq j$. Applying RI we get from $C(t_3) = c$ that $C(t_4) = c$. Also $C(t_2) = b$, implying $C(t_5) = b$, where $x(t_5, i) = b$, $x(t_5, j) = c$; $1 \leq j \leq k$; $i \neq j$ (applying RI on $C(t_2) = b$). Clearly $C(t_4) = c$ and $C(t_5) = b$ imply $b \sim_i c$. This is because $x(t_4, i) = c$, $x(t_4, j) = b$; $1 \leq j \leq k$; $i \neq j$ and $C(t_4) = c$ implies $C(t) = c$ with $x(t, i) = c, t \in \Gamma^k\{b, c\}$ (applying RI repetitively). Also $x(t_5, i) = b$, $x(t_5, j) = c$; $1 \leq j \leq k$; $i \neq j$ and $C(t_5) = b$ implies $C(t') = b$ with $x(t', i) = b, t' \in \Gamma^k\{b, c\}$ (applying RI repetitively). Thus $C(t) = x(r; t), \forall t \in \Gamma^k\{b, c\}$.

Also given $C(t_2) = b$, applying result of lemma 2, we get $C(t_3) \neq a$. Thus $C(t_3) = b$. We note that j is arbitrary. Thus applying RI we get choice from any tree in $\Gamma^k\{a, b, c\}$ will be b . What follows from this and RI is the following: $b \succ a$ and $b \succ c$. But this leads to contradiction, since we started by assuming $a \leftrightarrow b$ and $b \leftrightarrow c$.

Now, consider elementary k -ary trees formed with $\{a, b, c\}$, i.e. trees in the set $\Gamma^k\{a, b, c\}$. If $k = 3$, then consider $C(a, b, c)$. Since $a \leftrightarrow b$ and $b \leftrightarrow c$, from above discussion we cannot have either $C(a, b, c) = a$, $C(a, b, c) = b$ or $C(a, b, c) = c$. This is because in tree $t \equiv (a, b, c)$, each alternative appears only in single position. Thus if $k = 3$, we cannot have $a \leftrightarrow b$ and $b \leftrightarrow c$.

If $k = 4$, we consider t_1 and t_2 such that: $t_1 \equiv (a, b, c, a)$ and $t_2 \equiv (a, b, b, c)$. Consider $C(t_1)$ and $C(t_2)$, when $a \leftrightarrow b$ and $b \leftrightarrow c$. From above discussion, $C(t_1) = a$ and $C(t_2) = b$. This is because in t_1 , each of b and c occupies single position. Similarly, in t_2 , each of a and c occupies single position. Applying RI, we get from $C(t_1)$, $C(a, b, b, a) = a$ (*). Also applying RI to $C(t_2)$, we get $C(a, b, b, a) = b$ (**). We observe (*) and (**) contradict each other.

Now consider trees of order $k \geq 5$. Consider trees in $\Gamma^k\{a, b, c\}$ where first three positions from the left in the tree are occupied by c, b and a respectively and last $k - 3$ positions are all occupied by a but one position. Thus we consider following trees:

$$t_1 \equiv (c, b, \underbrace{a, a, a, \dots, a}_a, b)$$

$$t_2 \equiv (c, b, \underbrace{a, a, a, \dots, a}_a, b, a)$$

.

.

$$t_{k-3} \equiv (c, b, a, b, \underbrace{a, a, a, \dots, a}_a, a, a).$$

We first show $C(t_i) = a \forall i = 1, 2, \dots, (k-3)$ is not possible. We prove this by contradiction. Suppose $C(t_i) = a, \forall i = 1, 2, \dots, (k-3)$. Thus $C(c, b, a, b, a, a, a, \dots, a) = a$. Consider $C(c, b, a, b, c, a, a, a, \dots, a)$. This cannot be b . This is because if $C(c, b, a, b, c, a, a, a, \dots, a) = b$, then we have $C(c, b, a, b, a, a, a, \dots, a) = b$ by RI. Also $C(c, b, a, b, c, a, a, a, \dots, a) \neq c$. This is because if $C(c, b, a, b, c, a, a, a, \dots, a) = c$, then $C(c, b, a, a, c, a, a, a, \dots, a) = c$ by RI. But given above assumption, we have $C(c, b, a, a, b, a, a, a, \dots, a) = a$. This implies $C(c, b, a, a, c, a, a, a, \dots, a) = a$. This is contradictory to above $C(c, b, a, a, c, a, a, a, \dots, a) = c$. Thus we must have $C(c, b, a, b, c, a, a, a, \dots, a) = a$. Therefore $C(c, b, a, b, b, a, a, a, \dots, a) = a$ (*)

Consider $C(c, b, a, b, b, c, a, a, a, \dots, a)$. This cannot be c . This is because in that case we would have $C(c, b, a, a, a, c, a, a, a, \dots, a) = c$. But we have assumed $C(c, b, a, a, a, b, a, a, a, \dots, a) = a$. This implies $C(c, b, a, a, a, c, a, a, a, \dots, a) = a$. This contradicts $C(c, b, a, a, a, c, a, a, a, \dots, a) = c$. Also $C(c, b, a, b, b, c, a, a, a, \dots, a) \neq b$. This is because if $C(c, b, a, b, b, c, a, a, a, \dots, a) = b$, then by RI, $C(c, b, a, b, b, a, a, a, \dots, a) = b$. But this contradicts (*). Therefore we have $C(c, b, a, b, b, c, a, a, a, \dots, a) = a$. This implies $C(c, b, a, b, b, b, a, a, a, \dots, a) = a$ by RI...(**)

Continuing like this we can show $C(c, b, a, b, b, b, \dots, b, a) = a$. But we started with assumption $C(c, b, a, a, a, \dots, a, b) = a$. This implies $C(c, b, a, c, c, c, \dots, c, b) = c$. We can explain this as follows: $C(c, b, a, c, c, c, \dots, c, b) \neq a$, because a occupies only one position in this tree. Also $C(c, b, a, c, c, c, \dots, c, b) \neq b$, following lemma 2. Using RI we get $C(c, b, a, c, c, c, \dots, c, a) = c$. But using RI, we can show $C(c, b, a, b, b, b, \dots, b, a) = a$ implies $C(c, b, a, c, c, c, \dots, c, a) = a$. Thus we arrive at a contradiction.

Therefore $C(t_i) = a, \forall i = 1, 2, \dots, (k-3)$, is not possible. Thus $\exists t_j$, such that $C(t_j) = b$. W.l.o.g. we assume $t_j \equiv (c, b, a, a, a, a, b, a, a, \dots, a)$. Thus $C(t_j) = b$. But we must have $C(c, a, a, a, a, a, b, a, a, \dots, a) = a$. This is because both b, c occupy single position in this tree. Therefore by applying RI we have $C(c, a, a, a, a, a, c, \underbrace{a, a, a, \dots, a}_a) = a$ (***)

But we have $C(c, b, a, a, a, a, b, a, a, a, \dots, a) = b$. Hence $C(c, b, a, a, a, a, c, a, a, a, \dots, a) = c$. This follows from RI and lemma 2. This implies, by RI, $C(c, a, a, a, a, a, c, \underbrace{a, a, a, \dots, a}_a) =$

c. This contradicts (***) .

Hence we prove $a \leftrightarrow b$ and $b \leftrightarrow c$ cannot hold good.

■

Lemma 9 For any $a, b, c \in X$ if $a \sim_r b$ then it is not possible to have $b \leftrightarrow c$.

Proof: Suppose $a \sim_r b$, $b \leftrightarrow c$ for some $a, b, c \in X$. We consider possible binary relations between a and c :

◇ $a \succ c$: then $a \sim_r b$, $b \leftrightarrow c$ and $a \succ c$. Consider $C(t')$ where $t' \in \Gamma^k(\{a, b, c\})$ and $x_m(t') = a$, $x_n(t') = b$ ($m, n \neq r$) and $x_l(t') = c$, $\forall l (\neq m, n) \in \{1, 2, \dots, k\}$. $C(t') \neq c$ because if $C(t') = c$, then by RI we have $C(t'') = c$ where t'' is obtained by replacing b from n -th position by c in t' . But this contradicts $a \succ c$. Next we note that $C(t') \neq b$. This is because if $C(t') = b$, then by RI we have $C(t''') = b$ where t''' is obtained from t' by replacing c by a in all positions occupied by c in t' . But this contradicts $a \sim_r b$. Also $C(t') \neq a$. This is because if $C(t') = a$, then $C(t^*) = a$ where t^* is obtained by replacing c by b in all positions in t' that were occupied by b in t' . Thus $x_r(t^*) = b$. But this contradicts $a \sim_r b$.

All these prove that it cannot be that $a \sim_r b$, $b \leftrightarrow c$ and $a \succ c$.

◇ $a \sim_r c$: then $a \sim_r b$, $b \leftrightarrow c$ and $a \sim_r c$. Consider a tree $t' \in \Gamma^k(\{a, b, c\})$ such that $x_r(t') = b$, $x_s(t') = c$ ($r \neq s$) and $x_m(t') = a$, $\forall m \in \{1, 2, \dots, k\}$ ($m \neq r, s$). $C(t') \neq a$. This is because if it is true, applying RI we can show that $C(t'') = a$, where t'' is obtained by replacing c in t' by a . But this contradicts $a \sim_r b$. Thus we cannot have $a \sim_r b$, $b \leftrightarrow c$ and $a \sim_r c$.

◇ $a \leftrightarrow b$: then $a \sim_r b$, $b \leftrightarrow c$ and $a \leftrightarrow c$. In last lemma we have proved that it is never possible to have $x \leftrightarrow y$, $y \leftrightarrow z$, for any $x, y, z \in X$. Thus we can conclude it cannot be that $a \sim_r b$, $b \leftrightarrow c$ and $a \leftrightarrow c$.

Thus we can also conclude that it is not possible to have $a \sim_r b$, $b \leftrightarrow c$ for any $a, b, c \in X$.

■

Lemma 10 Consider any $a, b, c \in X$ such that $a \succ b$, $b \sim_r c$. Then $a \succ b$.

Proof: Suppose $a, b, c \in X$ such that $a \succ b$, $b \sim_r c$. Then $c \succ a$. Consider trees in $\Gamma^k(\{a, b, c\})$. In particular consider a tree $t' \in \Gamma^k(\{a, b, c\})$ such that $t \equiv (X^1, X^2, \dots, X^k)$,

where each X^i represents an elementary k -ary tree. Suppose for some arbitrary i ($1 \leq i \leq k$), a occupies only one arbitrary position in X^i , and other positions in X^i are occupied by b . Therefore suppose $x_r^i = a$, $x_s^i = b$; ($1 \leq r, s \leq k$, $r \neq s$). For each X^j ($1 \leq j \leq k$, $i \neq j$), we have following: $x_r^j = b$ and $x_s^j = c$, for all s ($1 \leq s \leq k$; $s \neq r$).

Consider $C(t)$. Given C satisfies CI and since $a \succ b$ and $b \sim_r c$, we have $C(t) = C(t')$. Here t' is an elementary k -ary tree, such that $x(t', i) = a$ and $x(t', l) = b$ for any l : $1 \leq l \leq k$, $l \neq i$. Since $a \succ b$, we have $C(t') = C(t) = a$.

We now replace b by c in each X^j (for any j : $1 \leq j \leq k$, $i \neq j$) of tree $t \equiv (X^1, X^2, \dots, X^k)$, and get tree t'' . Applying RI we get $C(t'') = a$. Again applying CI on $C(t'')$, and using $a \succ b$, we get $C(t'') = C(t''') = a$. Here t''' is an elementary k -ary tree with $x(t''', i) = a$ and $x(t''', l) = c$ for any l : $1 \leq l \leq k$, $l \neq i$. We observe in t''' , only one position (i -th position from the left) is occupied by a and this position is arbitrary. Thus we conclude $a \succ c$. ■

Lemma 11 Consider any $a, b, c \in X$ such that $a \succ b$, $b \leftrightarrow c$. Then $c \succ a$.

Proof: Suppose $a, b, c \in X$ such that $a \succ b$, $b \leftrightarrow c$. Consider trees in $\Gamma^k(\{a, b, c\})$. In particular consider a tree $t' \in \Gamma^k(\{a, b, c\})$ such that $t' \equiv (X^1, X^2, \dots, X^k)$, where each X^i represents an elementary k -ary tree. Suppose for some arbitrary i ($1 \leq i \leq k$), a occupies only one arbitrary position in X^i , and other positions in X^i are occupied by b . Therefore suppose $x_r^i = a$, $x_s^i = b$; ($1 \leq r, s \leq k$, $r \neq s$). Since $b \leftrightarrow c$, there exists an elementary tree (say, t_1), which comprises of only b and c , such that $C(t_1) = b$. This follows from definition of \leftrightarrow . Also suppose X^j , for any j ($1 \leq j \leq k$, $i \neq j$), represents elementary tree t_1 .

Consider $C(t)$. Given C satisfies CI and since $a \succ b$ and $b \leftrightarrow c$, we have $C(t) = C(t')$. Here t' is an elementary k -ary tree, such that $x(t', i) = a$ and $x(t', l) = b$ for any l : $1 \leq l \leq k$, $l \neq i$. Since $a \succ b$, we have $C(t') = C(t) = a$.

We now replace b by c in each X^j (for any j : $1 \leq j \leq k$, $i \neq j$) of tree $t \equiv (X^1, X^2, \dots, X^k)$, and get tree t'' . Applying RI we get $C(t'') = a$. Again applying CI on $C(t'')$, and using $a \succ b$, we get $C(t'') = C(t''') = a$. Here t''' is an elementary k -ary tree with $x(t''', i) = a$ and $x(t''', l) = c$ for any l : $1 \leq l \leq k$, $l \neq i$. We observe in t''' , only one position (i -th position from the left) is occupied by a and this position is arbitrary. Thus we conclude $a \succ c$. ■

We now state the main result of this section:

Theorem 2 *A choice function from k -ary trees ($k \geq 3$) satisfies CI and RI if and only if it is either PREF or PROC or SMEP.*

Proof: It is easy to show that PROC, PREF and SMEP satisfy RI and CI. In the other direction, suppose C is a choice function from trees in Γ^k , formed with alternatives from the set X . Also suppose C satisfies CI and RI. From lemma we note that C induces either of following binary relations over set X : \succ , \sim_r ($1 \leq r \leq k$) or \leftrightarrow . Thus these binary relations are complete over X . Based on above lemmas we further consider following:

Suppose $a, b \in X$ and $a \succ b$. Given this, we consider binary relation between b and any other alternative $c \in X$. we consider following possibilities: (i) $b \succ c$; (ii) $c \succ b$; (iii) $b \sim_r c$; (iv) $b \leftrightarrow c$. If (i) holds good we have $a \succ c$ by transitivity of \succ (lemma 4). Similarly if (iii) holds, we have $a \succ c$ by lemma 5 . Also in case (iv) holds we have $a \succ c$ by lemma 6 . In case (ii) holds, we can have either $a \succ c$ or $c \succ a$. We cannot have $a \sim_r c$ (from lemma 8 we cannot have $a \sim_r c$ and $c \succ b$) or $a \leftrightarrow c$ (from lemma 9 we cannot have $a \leftrightarrow c$ and $c \succ b$).

Suppose $a, b \in X$ and $a \sim_r b$ ($1 \leq r \leq k$). Given this, we consider binary relation between b and any other alternative $c \in X$. we consider following possibilities: (i) $b \succ c$; (ii) $c \succ b$; (iii) $b \sim_r c$; (iv) $b \sim_m c$ ($1 \leq m \leq k$ and $m \neq r$); (v) $b \leftrightarrow c$. From lemma 9 we understand that (i) cannot hold. From lemma 5 we have (ii) implies $c \succ b$. Transitivity of \sim_r (lemma 4) implies that $a \sim_r c$, if (iii) holds good. From lemma we cannot have (iv). (v) is also impossible following lemma 11.

Suppose $a, b \in X$ and $a \leftrightarrow b$. Given this, we consider binary relation between b and any other alternative $c \in X$. we consider following possibilities: (i) $b \succ c$; (ii) $c \succ b$; (iii) $b \sim_r c$; (iv) $b \leftrightarrow c$. From above lemmas we observe only (ii) can hold good and $c \succ a$ also holds good.

Therefore what follows from above is the following: if for two alternatives $a, b \in X$, we have $a \leftrightarrow b$, then for any other alternative $c \in X$ we can only have $c \succ a$ and $c \succ b$. Also for any other alternative $d \in X, d \neq a, b, c$ we can only have $d \succ c$ or $c \succ d$. On the whole we get $X' \subset X$ such that $X' = \{a, b\}$ and for any two alternatives $x_1, x_2 \in \{X - X'\}$ we have (i) either $x_1 \succ x_2$ or $x_2 \succ x_1$; and (ii) $x_1 \succ a, x_2 \succ b$. Clearly this constitutes a Single Maximal Element Preference over X . Now consider choice from any k -ary tree $t \in \Gamma^k$. Given a single maximal element preference over X , we can apply CI and get $C(t) = M(X(t), R)$. If $|X(t)| > 2$ then $M(X(t), R)$ must contain a unique element following R is single maximal element preference ordering. But if $|X(t)| = 2$ and if for $\{a, b\} \in X(t)$, $a \leftrightarrow b$, then $C(t)$ is obtained applying choice function induced by \leftrightarrow (C_{\leftrightarrow}) to all elementary trees which are horizontally concatenated to form some k -ary trees and which are again concatenated to form some k -ary trees and continues like this to finally form the tree t . Recursive use of choice

function C_{\leftrightarrow} to all stages leads to choice from final elementary k -ary tree and x results from applying C_{\leftrightarrow} to this tree. Thus C is *SMEP*.

If for two alternatives $a, b \in X$, we have $a \sim_r b$, then for any other alternative $c \in X$ we can have either $c \succ a$ (implying $c \succ b$) or $c \sim_r a$ (and thus $c \sim_r b$). Thus either all the alternatives in X are related to each other by \sim_r (and therefore we can note C is *PROC* which follows from C satisfying CI and all the alternatives related by \sim_r .) or we have following: there is a subset $X' \subset X$ ($a, b, \in X'$) such that all alternatives in X' are related to each other by sim_r and for any $x \in \{X - X'\}$ and any $y \in X'$, we have $x \succ y$. Also for any $x_1, x_2 \in \{X - X'\}$ we have $x_1 \succ x_2$ or $x_2 \succ x_1$. Thus in the last case we have a Single Maximal Element Preference and C is *SMEP*.

After considering above possibilities only one possibility is left out: alternatives are related to each other by an antisymmetric, complete and transitive order \succ . Thus C is **PREF**.

Thus choice function C from k -ary trees and satisfying RI and CI must either be **PREF** or **PROC** or **SMEP**.

■

6 CHOICE FROM MIXED TREES

In this section we briefly discuss choice functions from *Mixed Trees*, when choice functions satisfy CI and RI axioms. Therefore domain for choice functions is Γ . Also by $\Gamma(B)$ we denote the set of all possible trees formed with all alternatives from set B , $B \subset X$.

We first give some definitions:

Definition 25 Suppose C is a choice function from trees in Γ . We define a binary relation $\tilde{\sim}_r$ over set X as follows: $a \tilde{\sim}_r b$ if and only if for any tree $t \in \Gamma_e\{a, b\}$, we have $C(t) = x(r; t)$, $1 \leq r \leq 2$.

Definition 26 A binary relation R over set of alternatives X is called **Restricted Single Maximal Element Preference** if :

◇ R is complete and transitive.

◇ $\exists X' \subset X$ such that for any $x, y \in X', x \neq y$ we have either xRy or yRx but not both, i.e, R is antisymmetric over X' . Also for any $m, n \in \{X - X'\}$, we have mRn and nRm ,

i.e. R is symmetric over $\{X - X'\}$. In addition, for any $x \in X', m \in \{X - X'\}$, we have xRm but $-mRx$.

◇ for any $m, n \in \{X - X'\}$, we have $m \sim_r n; 1 \leq r \leq 2$, i.e. alternatives in $\{X - X'\}$ are related to each other by $\sim_r; 1 \leq r \leq 2$.

Definition 27 A choice function C from trees in Γ is **Restricted Single Maximal Element Preference** based or **RSMEP**, if there exists a Restricted Single Maximal Element Preference R , such that, for any $t \in \Gamma$, we have $C(t) = M(X(t), R)$. If $M(X(t), R)$ is not singleton, i.e. for any $x, y \in M(X(t), R)$ we have $x \sim_r y; 1 \leq r \leq 2$, then $C(t) = x(r; t)$.

Definition 28 A choice function C from trees in Γ is **Mixed Order** based or **MO** if for any $t \in \Gamma$, we have $C(t) = x, x \in X(t)$ such that following holds: Suppose $t \equiv (X^1, X^2, \dots, X^M)$. For each set of k -ary trees ($k \geq 2$), we have set of given choice functions: for binary trees we have a **Mixed** choice function, for ternary or higher order trees we have either a **PREF** or a **PROC(r)** or a **SMEP**. Only restriction is following: suppose aR_1b, bR_1c , where R_1 is a weak order or pure preference or single maximal element preference, such that R_1 is anti-symmetric between a and c and is induced by some choice function C_1 . Then for some other choice function C_2 , which induces an ordering R_2 , it cannot be that cR_2a . Then if X^1 is a binary tree, then the Mixed choice function is applied to it or if it is a higher order tree, we apply given choice functions. We follow this for each X^i ($1 \leq i \leq M$). We thus reduce the tree using given choice functions at each non-terminal node, depending on the order of the tree emanating from that node. Continuing this we get x .

Example 6 Consider set of alternatives $X = \{a, b, c, d, e, f\}$. Consider a choice function C from mixed trees, such that C is a MO. Suppose C is parameterized with choice functions C_2, C_3, C_4 etc., where C_2 is choice function from binary trees and C_3 is a choice function from ternary trees and so on. Now consider $C(t)$, where t is given in next figure.

Suppose C_4 is a PREF and is characterized by pure preference R_4 , so that $cR_3d, dR_3a, aR_3b, bR_3e, eR_3f$. Suppose C_3 is PROC(1). Thus here $C(t) = a$.

The main result of this section is given below:

Theorem 3 A choice function C from mixed trees satisfies CI and RI if and only if either of the following holds good:

(i) C is PREF;

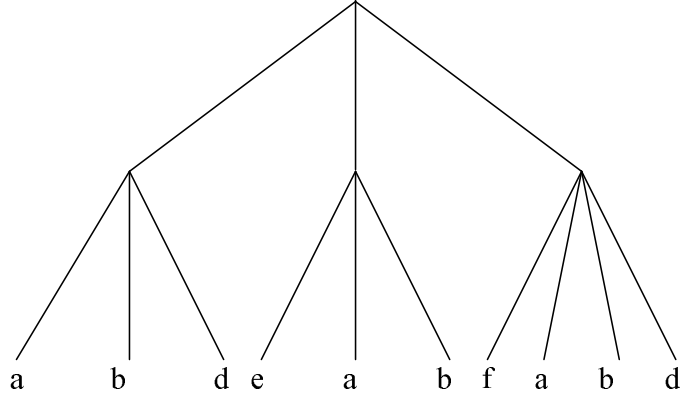


Figure 17: Tree: t

(ii) C is $PROC(1)$ or $PROC(2)$ or EXT ;

(iii) C is $RSMEP$;

(iv) C is MO .

Proof: Suppose C is a choice function from trees in Γ and C satisfies CI and RI. Since $\Gamma^k \subset \Gamma$, we note characterizations for choice functions from k -ary trees ($k = 2, 3, \dots$) hold good for choice from Γ^k ($k = 2, 3, \dots$). Given C satisfies CI, we can show that it must either be one of (i) - (iv). [To be completed].

■

REFERENCES

- [1] Ordeshook, P.C., Schwartz, T., 1987. Agendas and the control of political outcomes. *The American Political Science Review* 81, 179-200.
- [2] Plott, C.R., 1973. Path independence, rationality, and social choice. *Econometrica* 41, 1075-1091.
- [3] Rubinstein, A., Salant, Y., 2006. A model of choice from lists. *Theoretical Economics* 1, 3-17.
- [4] Salant, Y., Rubinstein, A., 2007. (A, f) Choice with frames. *Review of Economic Studies* 75, 1287-1296.