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Foundations of Spatial Preferences

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Foundations of Spatial Preferences*

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Abstract

I provide an axiomatic foundation for the assumption of specific utility functions in a multidimensional spatial model, endogenizing the spatial representation of the set of alternatives. Given a set of objects with multiple attributes, I find simple necessary and sufficient conditions on preferences such that there exists a mapping of the set of objects into a Euclidean space where the utility function of the agent is linear city block, quadratic Euclidean, or more generally, it is the δ power of one of Minkowski's [25] metric functions. In a society with multiple agents, I characterize the set of preferences that are representable by weighted versions of a family of functions that are indexed by a parameter $\delta > 0$, where $\delta \geq 1$ corresponds to the set of Minkowski's functions. In light of the starkly different consequences between representability with $\delta \leq 1$ or with $\delta > 1$, I propose a test to empirically estimate δ.

JEL Classification: D81, D72.

Keywords: Utility representation, spatial models, multidimensional preferences, spatial representation.

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Consider objects that have multiple attributes, and values within each attribute have a natural order. A multidimensional spatial model is useful to analyze preferences and choices over these objects. The original spatial model was presented by Hotelling [13] to study product differentiation in the real line. David, DeGroot and Hinich [26] adapted the multidimensional spatial model to study political competition over multiple policy issues, letting each policy issue correspond to a dimension in a vector space. The standard approach in political economy is to assume that each agent has an ideal policy bundle represented by a most preferred point in the vector space, and that preferences over policy bundles are representable by a utility function that decreases in the Euclidean distance to the ideal point of the agent.

In many applications, attributes are not objectively quantifiable and the choice set is not a subset of a vector space. For instance, policies on social values or foreign policy do not have natural units of measure. Policies can be represented as vectors, but this representation is subjective and endogenous, and any assumption on the utility of an agent that depends on an endogenous spatial representation and not on the original set of alternatives is suspect, because the same preferences can be represented by utility functions of very different shapes depending in another spatial representation of the set of alternatives into points in a vector space.

In this paper I characterize the set of preferences over alternatives that are representable by city block utility functions, by Euclidean utility functions, or more generally, by weighted generalizations of Minkowski's [25] metric functions. I highlight that the map of the set of alternatives into a vector space is just a representation, much like a utility function is a representation of preferences, and both the spatial mapping and the utility function on this space are endogenously chosen for their convenient role as objects that are more tractable than a binary relation over the primitive set of alternatives. The characterizations provide axiomatic micro-foundations for the existing multidimensional spatial models in the literature, improving our understanding about their assumptions.

It is a common practice to assume that the shape of the indifference curves is given by the Euclidean distance, to assume that the utility function is quadratic or more generally strictly concave, and to interpret quadratic loss or concave utilities as risk aversion.[1]

The distance function determines the shapes of the indifference curves. The substantive implications of the chosen shapes are great. If preferences are Euclidean or more generally smooth, for a generic distribution of ideal points the core of simple majority is generically empty and majority rule is globally intransitive.[2] On the other hand, if preferences decrease in the city block distance, then under more general conditions the majority rule core is not empty and there exists a stable policy outcome.[3]

However, the shape of the indifference curves depends on the spatial representation of alternatives. Given any preferences, there is a spatial representation of alternatives such that utility functions that represent these preferences are not smooth and the results on the emptiness of the core and the intransitivity of majority rule do not apply. The more basic question is to identify the set of preferences that can be represented by smooth utility functions in some spatial map of alternatives. These preferences, whether we represent them as smooth utility functions in some map or as non-smooth utilities in a different map, generate intransitivities and an empty core with simple majority rule.

Similarly, if the set of alternatives does not have an exogenous spatial representation, the question of whether the utility function should be concave or convex is moot: The same individual preferences can be represented at wish as risk averse, risk loving or risk neutral by distorting the spatial location of the alternatives. Unless the policy space is exogenously defined, the convexity of the utility function does not have an

[1]See for instance Enelow and Hinich [9], Feddersen [10] or Schofield and Sened [32] for a small sample of highly cited work that uses quadratic Euclidean preferences.

[2]See Plott [27], McKelvey and Schofield [22] and McKelvey [20] and [21].

[3]See Rae and Taylor [29], Wendell and Thorson [36], McKelvey and Wendell [23] and Humphreys and Laver [14].

interpretation in terms of the preferences of the agent over uncertain outcomes or her attitude toward risk and lotteries. It is, in fact, a joint assumption on the preferences over lotteries and the chosen spatial representation of the set of alternatives, and the terms "risk aversion" or "risk neutrality" are not meaningful.

The implication is that to the extent that the spatial representation is an object of choice for theorists and not a primitive object, any assumption over preferences on this vector space is difficult to interpret. It is preferable to make assumptions on the primitives of the choice problem: on the original set of alternatives, which is not exogenously endowed with any spatial representation, and the preferences over this set. This is exactly what I do in this paper.

I study preferences over a set of alternatives with multiple attributes. Each attribute is endowed with a natural order, but not with a notion of distance between different values in this attribute. Agents have preferences over lotteries over alternatives. In a political economy application, an alternative is a policy bundle, an attribute is an issue and policies on a given issue are ordered, but they are not exogenously quantified. For instance, while legalization and illegalization of abortion in all cases lie at opposite ends of feasible policies on the issue of abortion, the exact location of any intermediate policy in $[0, 1]$ is an endogenous choice on the part of the researcher who wishes to study choices not over the primitive set of objects, but over a more convenient spatial representation of these objects into a vector space.

I assume that preferences are representable by a utility function. I characterize the set of preferences such that there exists a mapping of alternatives into a Euclidean space where each dimension corresponds to an attribute, the location of values along each dimension in the space is monotonic in the given order within each attribute, and the preferences over points in the Euclidean space are representable by a utility function that decreases in the δ power of a distance function $d^{\delta}(y, y^{i})$ to an ideal point y^{i}, where $d^{\delta}(y, y^{i})$ belongs to Minkowski's [25] family of d^{δ} distances, $d^{\delta}(y, y') = \left(\sum_{k} |y_k - y'_k|^{\delta}\right)^{1/\delta}$. Linear city block preferences or quadratic Euclidean preferences are particular cases with $\delta = 1$ and $\delta = 2$. I show that the necessary and sufficient conditions are the same for any δ. Preferences must be multi-attribute single peaked and modular. Multi-attribute single peakedness is an extension of the standard notion of single peakedness, so that preferences are single peaked on any given attribute. Modularity is a separability condition. Preferences are modular if an agent evaluates attributes independently of each other, so that her preference over one attribute is invariant with changes in other attributes.

It may appear surprising that preferences are representable by a linear city block utility function if and only if they are representable (in a different space) by a quadratic Euclidean utility function, given the positive results on existence of core outcomes if utilities are city block, and the negative results if utilities are smooth. There is no paradox: The results on core existence require all agents to have city block utility functions given a common spatial representation of the set of alternatives; it does not suffice that each agent has city block utilities given her own idiosyncratic map of alternatives into a vector space. The additional requirement of a common spatial representation breaks down the identity of the sets of preferences representable by linear city block or quadratic Euclidean utilities, imposing additional conditions on preferences that now depend on the desired distance function. I state and explain the additional conditions for a utility function that is linear in a weighted city block distance, for a utility function that is quadratic in a weighted Euclidean distance, and, more generally, for utility functions that decrease in the δ power of a weighted version of a d^{δ} distance. The condition for a linear city block utility function is simpler and I argue that it is more intuitive as well; to check if this intuition is good, I propose an empirical test to estimate in laboratory experiments which of the conditions is more likely to approximately hold.

While the main application I use for motivation and intuition for my results is political competition over multiple issues, where each attribute is a policy issue, and each alternative is a policy bundle that can be represented as a point in the vector space, multidimensional spatial models are also used in industrial organization theories of product differentiation, where a good or product is described as a collection of characteristics, so that each characteristic is an attribute, and the good is an alternative that can be represented as a point in a vector space.[4]

To my knowledge, the directly related literature on the representability of preferences over multidimensional objects with an endogenous spatial representation is scant. D'Agostino and Dardanoni [7] characterize the Euclidean distance function in terms of five invariance and monotonicity axioms, Kannai [16] and Richter and Wong [30] find conditions such that preferences in a given space can be represented by a concave utility function, Kalandrakis [15] investigates whether the incomplete preferences revealed by a finite number of binary voting choices is consistent with a concave utility representation of these preferences, and Degan and Merlo [8] question whether the hypothesis that voters vote according to a utility function that is decreasing in the Euclidean distance is empirically falsifiable when the ideal point of the voter is unknown.

The closest reference is by Bogomolnaia and Laslier [4]. They find how many dimensions must be used to represent any ordinal preference profile over a finite number of alternatives using Euclidean preferences. Because their set of alternatives is finite, for a single individual, their problem is trivial. Any preference can be represented in just one dimension by assigning alternatives to natural numbers according to the preference order of the agent. By contrast, I consider an infinite number of alternatives by studying lotteries over alternatives. A more substantive difference is that in their theory, alternatives are not defined as a collection of attributes, and hence the dimensions of the space are fully endogenous. My motivation to consider alternatives with multiple attributes is political competition over multiple political issues, issues such as income taxation, public health care provision, or immigration. I treat each of this issues as an attribute, so that an alternative is a policy bundle with a policy pre-

[4]See Gorman [12], Lancaster [19] and Rosen [31] for seminal papers, Tirole [33] for a textbook overview, and Caplin and Nalebuff [5] and Berry and Pakes [2] for more advanced developments.

scription on each issue. I seek to find an spatial representation in K dimensions that is consistent in each dimension with the natural order of values within each of the K exogenously given attributes. Since the problem I address has more restrictions, not every preference relation is representable in any space using the city block distance or Euclidean distances, even if there is a single agent. I find axiomatic conditions on the preference relation under which, in some space, it is representable by a utility function that depends on the desired distance.

1 The Theory

Let A be a set of attributes, of size K. For each attribute $k \in A = \{1, ..., K\}$, let X_k be the set of possible values on attribute k. This set can be finite, countable or uncountable. Let the elements of X_k be ordered by a linear order $\geq_k$ with a maximal element $x_k^{\max}$ and a minimal element $x_k^{\min}$. Let $>_k$be the strict order derived from $\geq_k$. Given the set of possible values on each attribute, let the set of alternatives be the Cartesian product $X = X_1 \times X_2 \times ... \times X_K$ and let ΔX be the set of simple lotteries on X. In a political economy application, each attribute $k \in A$ is a policy issue and X is the set of alternative policy bundles.

For any given lottery $p \in \Delta X$, let $p(x)$ denote the probability that p assigns to $x \in X$. For any $p \in \Delta X$, let $supp(p) = \{x \in X : p(x) > 0\}$ be the support of p, the subset of alternatives to which lottery p assigns positive probability. Slightly abusing notation, let $x, y, z, w \in X$ denote as well degenerate lotteries, so they belong to ΔX. Let x_k denote the $k - th$ element of the ordered list x, let $X_{-k} = X_1 \times ... \times X_{k-1} \times X_{k+1} \times ... \times X_K$ and let $x_{-k} \in X_{-k}$ denote the ordered list of length $K - 1$ that contains all attribute values of alternative x except x_k. Then we can write x as $x = (x_k, x_{-k})$. Let $p_k \in \Delta X_k$ be a simple lottery on X_k, let $p_k(x_k)$ the probability that p_k assigns to x_k and let $(p_k; x_{-k}) \in \Delta X$ be the lottery over alternatives that runs lottery p_k on attribute k and yields x_{-k} with certainty in all other attributes.

Let $\succsim$ be a complete and transitive binary relation on ΔX representing the weak preferences of agent i over lotteries on X. Let $x \succ y$ denote ($x \succsim y$, not $y \succsim x$) and let $x \sim y$ denote ($x \succsim y$, $y \succsim x$). Let $\succsim$ satisfy the independence and archimedean axioms due to Von Neumann and Morgenstern [35].

Axiom 1 *(Archimedean): If $p, q, r \in \Delta X$ such that $p \succ q \succ r$, then there is an $\alpha \in (0, 1)$ such that $\alpha p + (1-\alpha)r \sim q$.*

Axiom 2 *(Independence): For all $p, q, r \in \Delta X$ and any $\alpha \in (0, 1)$, then $p \succsim q$ if and only if $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.*

Then the preferences over lotteries can be represented by a utility function $u : X \longrightarrow \mathbb{R}$ such that for any $p, q \in \Delta X$, $p \succsim q$ if and only if $\sum_X p(x)u(x) \geq \sum_X q(x)u(x)$. This is part of the celebrated expected utility theorem by Von Neumann and Morgenstern.

A spatial representation of X is a vector valued function $f = (f_1, f_2, ..., f_K)$ such that $f_k : X_k \longrightarrow \mathbb{R}$ is strictly increasing in $\geq_k$ for each $k \in A$ and $f(x) \in \mathbb{R}^K$ represents alternative $x \in X$. Let $\mathcal{F}$ be the set of all possible spatial representations satisfying this monotonicity requirement. The motivating question is under what conditions on $\succsim$ there exists a spatial representation f such that the preferences over $f(X) \subseteq \mathbb{R}^K$ can be represented according to a given utility function.

Let $L(x, y)$ be a lottery that assigns equal probability to x and y. Let $x \vee y = (\max\{x_1, y_1\}, ..., \max\{x_K, y_K\})$ and $x \wedge y = (\min\{x_1, y_1\}, ..., \min\{x_K, y_K\})$ be the join and the meet of x and y.

Axiom 3 *(Modularity) For all $x, y \in X$, $L(x, y) \sim L(x \vee y, x \wedge y)$.*

Modular preferences are such that the agent evaluates changes in one attribute in the same manner, regardless of the values in other attributes. For added intuition, consider an example with two issues and let x, y lie in the non positive quadrant with respect to the ideal policy, so that among all four options in the lotteries, $x \vee y$ is the best outcome in both issues, $x \wedge y$ is the worst outcome in both issues, and x and y are each good in one issue and bad in the other. If the outcome is determined by two lotteries, one on each issue, and these lotteries assign equal probability to the good and bad outcome on their respective issue, an agent with modular preferences is indifferent about the correlation of the two lotteries. Birkhoff [3] calls a function f satisfying $f(x)+f(y) = f(x \vee y)+f(x \wedge y)$ a *valuation.* In two dimension, modularity is equivalent to the standard separability condition by Fishburn [11], theorem 11.1. With more than two dimensions, modularity is simpler and a (negligibly) weaker assumption: Fishburn's separability implies modularity, and modularity together with transitivity implies Fishburn's separability. See Kreps [17], Milgrom and Shannon [24] and Topkis [34] for related ordinal and cardinal definitions of modularity.

Axiom 4 *(Multi-attribute single peakedness)* $\exists x^* \in X$ *such that for each* $k \in \{1, 2, ..., K\}$, *and any* $x_k^1, x_k^2, x_k^3, x_k^4 \in X_k$:

$$x_k^1 \leq_k x_k^2 \leq_k x_k^* \leq_k x_k^3 \leq_k x_k^4 \Longrightarrow (x_k^2, x_{-k}^*) \succ (x_k^1, x_{-k}^*) \text{ and } (x_k^3, x_{-k}^*) \succ (x_k^4, x_{-k}^*).$$

A multi-attribute single peaked preference relation has a best policy such that, moving away from the peak on any given attribute, preferences decrease, as in a unidimensional single peaked relation. This condition of single-peakedness is weaker than the multi-dimensional single peakedness used by Barberà, Gul and Stacchetti [1], but together with modularity, it suffices to guarantee that their stricter restriction is also satisfied, and that alternatives and preferences can be represented in a vector space such that the utility of the agent is a decreasing function of any of Minkowski's [25] norms.

Theorem 1 *Suppose* $\succsim$ *is representable by the expected utility of* $u : X \longrightarrow \mathbb{R}$. *For any* $\delta \in (0, \infty)$, *a spatial representation* $f^\delta = (f_1^\delta, ..., f_K^\delta) \in \mathcal{F}$ *such that*

$$u(x) = -\sum_{k=1}^{K} |f_k^\delta(x_k)|^\delta$$

exists if and only if $\succsim$ *is multi-attribute single peaked and modular.*

This and all other proofs are in the appendix. Most relevant in applications, theorem 1 says that if preferences are modular and multi-attribute single peaked, we can represent alternatives and preferences in a specific vector space fixing the ideal alternative of the agent at the origin of coordinates and using a utility function that is linear in the city block norm, or we can represent them in a different space using a utility function that is quadratic in Euclidean norm. More generally, for any $\delta > 0$, let $d^{\delta}(y, y') = \left(\sum_{k=1}^{K} |y_k - y'_k|^{\delta} \right)^{1/\delta}$. Note that if $\delta \geq 1$, the function $d^{\delta}(y, y')$ is a Minkowski [25] metric, whereas if $\delta < 1$, it is not a metric because it violates triangle inequality. I nevertheless refer to it as a distance function in an informal use of the term *distance* because for a fixed ideal point y', the function gives an intuitive notion of proximity to this ideal point. I reserve the term *metric* for the mathematical notion of distance satisfying symmetry and triangle inequality. For any $\delta > 0$, mapping the most preferred alternative of the agent to the origin of coordinates and choosing the appropriate spatial representation $f^{\delta}(X)$, we can decompose the utility function $u(x) = l \circ d \circ f^{\delta}(x)$, where the distance function $d(\cdot)$ is equal to $d^{\delta}(f^{\delta}(x), 0)$ and the loss function $l(\cdot)$ is a power function of degree δ. In figure 1 I illustrate the indifference curves with $\delta = 0.5$, $\delta = 1$, $\delta = 2$, and $\delta = 4$.

What we cannot do is represent the preferences in any space using a utility function that is linear or exponential in the Euclidean distance, such as the one used in the celebrated D-NOMINATE method to estimate the location of the ideal policy in two dimensions of US legislators devised by Poole and Rosenthal [28]. Euclidean utility functions that are not quadratic in the Euclidean distance are inconsistent with preferences satisfying the modularity assumption. I state this as a more general claim that follows as a corollary from the proof of theorem 1.

Claim 2 *Suppose* $\succsim$ *is modular and representable by the expected utility of a function* $u : X \longrightarrow \mathbb{R}$. *Suppose that there exist a spatial representation* $f = (f_1, ..., f_K) \in \mathcal{F}$

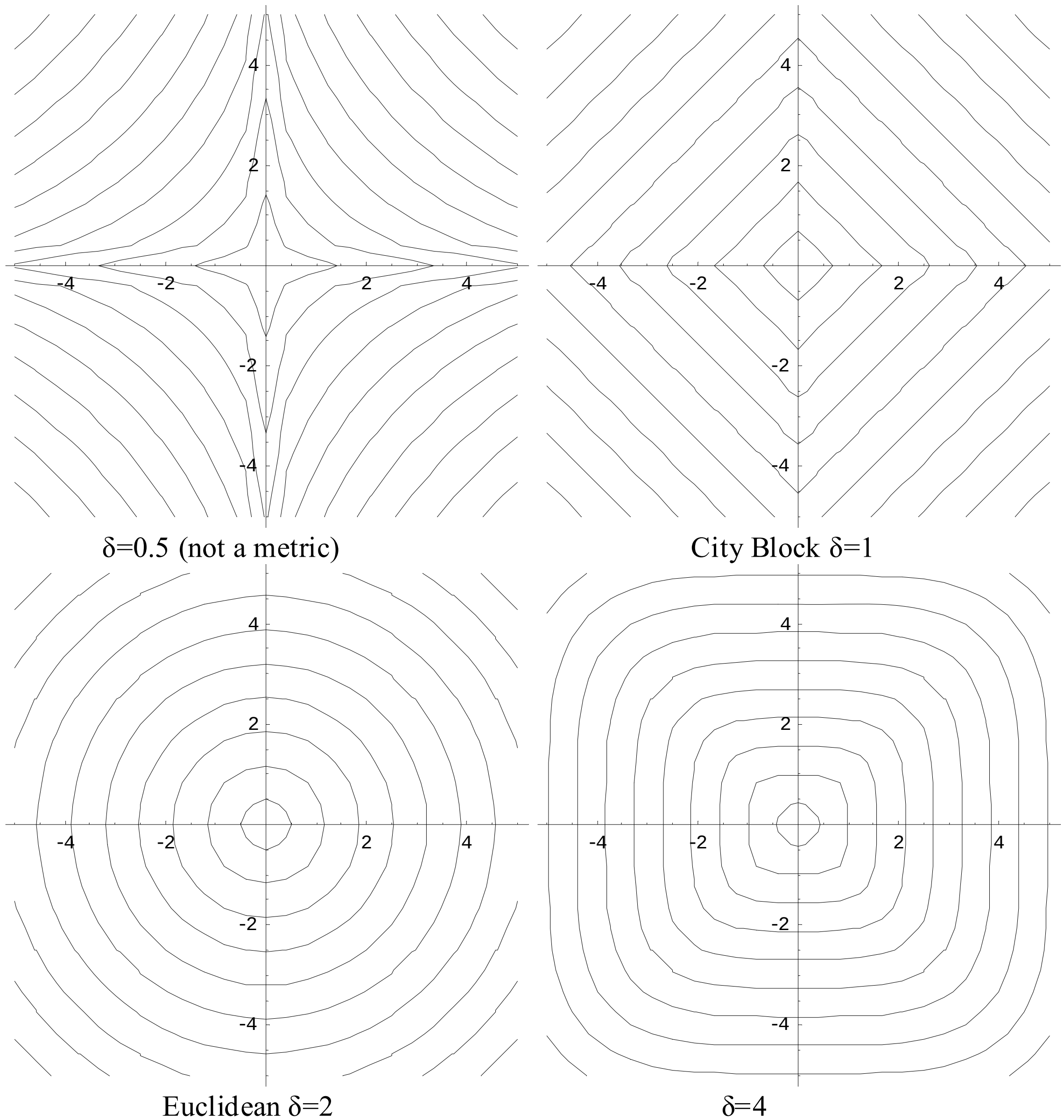


Figure 1: Indifference curves given by the distance function $d^{\delta}(y,0)=\left(\sum_{k=1}^{K}|y_k|^{\delta}\right)^{1/\delta}$.

and a strictly increasing loss function $l : \mathbb{R} \longrightarrow \mathbb{R}$*, such that*

$$u(x) = -l\left(\left(\sum_{k=1}^{K} |f_k(x_k)|^{\delta}\right)^{1/\delta}\right) \text{ for some } \delta \in \mathbb{R}_{++}.$$

Then $l(d) = \alpha + \beta d^{\delta}$ *for some* $\alpha, \beta \in \mathbb{R}$.

The parameters α, β merely note that utility functions are unique only up to affine transformations; normalize $\alpha = 0$ and $\beta = 1$ for the simplest expression. Modularity is a separability assumption that requires agents to treat issues independently, assessing their preferences over policies on one issue (or over lotteries over policies on one issue) in the same manner regardless of the policies in any other issue. Whether preferences are separable across issues is an empirical question. Lacy [18] searches for evidence of non separability across pairs of issues that seem to be related, such as taxes and spending, pollution regulation and cleaning up of the environment, or the status of English as an official language and immigration laws. He finds mixed evidence. While outside his study, I conjecture that most agents have separable preferences across issues that do not seem to be related, such as the status of English as an official language and environmental protection.

Theorem 1 makes explicit the restriction on primitives implied by the use of the city block or quadratic Euclidean utilities. Interestingly, the implicit restriction is the same for these two commonly used utility functions: Preferences over the primitive set of alternatives must satisfy separability in the sense of modularity, and single peakedness. To my knowledge, there is no parallel characterization of the set of preference profiles over the primitive set of alternatives that are consistent with a utility function that is linear or exponential in the Euclidean distance. Bogomolnaia and Laslier [4] show that any preference relation can be represented by Euclidean preferences in a space with a sufficiently large number of dimensions, but the assumption of linear or exponential Euclidean preferences in a given space with a fixed, small number of dimensions implies unknown and possibly unwarranted restrictions on the admissible preferences defined over the primitive set of alternatives.

As I discuss in the introduction, it may seem surprising that the same preference relation can be represented using a city block utility function, or using a quadratic Euclidean utility function, particularly in light of the results on generic inexistence of majority voting core outcomes if preferences are Euclidean and the more positive results on the existence of core outcomes under majority voting with city block preferences, but the results on existence of core outcomes rely on a common space for all agents in a society with at least three agents. The need for a common spatial representation imposes further restrictions that I detail in the next section.

2 A Common Space for Multiple Agents

In the previous section, the primitive on preferences is a complete and transitive binary relation $\succsim$ on ΔX that satisfies the archimedean and independence axiom, so that $\succsim$ is representable by the expected utility of a utility function defined over X.

In this section I extend theorem 1 to a society with multiple agents. For explanatory purposes I first focus on the city block case $\delta = 1$ in proposition 3, and the quadratic Euclidean case $\delta = 2$ in proposition 4.

In a society N with n agents, the new primitive are n complete and transitive binary relations defined on ΔX that satisfy the archimedean and independence axiom, so that $\succsim_i$ is representable by the expected utility of a utility function u_i defined over X for any $i \in N = \{1, ..., n\}$, and I also assume that each preference $\succsim_i$ satisfies modularity and multi-attribute single peakedness. The challenge in this section is to find the additional conditions so that the preferences of every agent are representable by the desired utility functions with a unique spatial representation common to all agents. Let $\succsim_N \equiv (\succsim_1, ..., \succsim_n)$ and assume that each agent i has a unique preferred alternative denoted $x^i \in X$ so that $x^i \in \Delta X$ is the maximal element of the order $\succsim_i$.

Agents may care more about some attributes than others. With a single agent, this is easily solved by appropriately rescaling the units of the spatial representation of the alternatives. With multiple agents, it is necessary to introduce weights for each dimension. Given any $y, y' \in \mathbb{R}^K$, the standard l_1 norm is $||y||_1 = \sum_{k=1}^{K} |y_k|$ with its associated l_1 metric $||y - y'||_1$. The second graph in figure 1 depicts this distance in $\mathbb{R}^2$. I consider a generalization that assigns different weights to each dimension, and to each direction away from the ideal point on each dimension. Allowing the weights on each dimension not to be a constant, but to be a function of the side of the half space given by the ideal value of the agent in this dimension, the distance functions are no longer a mathematical metric, since they violate symmetry. I nevertheless refer to them as distances, consistent with the intuition that they measure the separation or difficulty to travel from a point to another. For a purely geographical interpretation, if A is a point uphill and B is a point downhill, the walking distance from A to B is less than the walking distance from B to A.

Definition 1 *Given any vector of weights* $\lambda \equiv (\lambda_{1+}, \lambda_{1-}, \lambda_{2+}, \lambda_{2-}, ..., \lambda_{K+}, \lambda_{K-}) \in \mathbb{R}_+^{2K}$, *a generalized weighted city block distance with weights* λ *is a function* $g : \mathbb{R}^K \times \mathbb{R}^K \longrightarrow \mathbb{R}_+$ *such that for any* $y, y' \in \mathbb{R}^K$,

$$g(y, y') = \sum_{k=1}^{K} \lambda_k(y_k, y'_k)|y_k - y'_k|, \text{ where } \lambda_k(y_k, y'_k) = \{\lambda_{k+} \text{ if } y_k \geq y'_k,\ \lambda_{k-} \text{ if } y_k < y'_k\}\ \forall k \in A.$$

Given agent $i's$ ideal point $y^i \in \mathbb{R}^K$, generalized weighted city block preferences are generated in the standard way, letting y be preferred to y' if and only if y is closer than y' to y^i. Weighted city block preferences in $\mathbb{R}^2$ have indifference curves that are rhombi with diagonals along the axes. Allowing weights to differ on either side of the ideal point creates indifference curves that in $\mathbb{R}^2$ are tangential quadrilaterals with diagonals along the axes, but sides of unequal length. See figure 2 (left).

Since the space of alternatives X is not originally contained in $\mathbb{R}^K$, the distance is defined over the mapping $f(X)$ of alternatives into an Euclidean space, so given her ideal object x^i, agent i has generalized weighted city block preferences over $f(X)$ if for any $x, x' \in X$, agent i prefers x to x' if and only if the generalized weighted city block distance from $f(x)$ to $f(x^i)$ is smaller than the generalized weighted city block distance from $f(x')$ to $f(x^i)$.

Given any attribute $k \in A$, let $l_k, h_k \in N$ be such that $x_k^{l_k} \leq_k x_k^i \leq_k x_k^{h_k}$ $\forall i \in N$. These are the agents with a lowest and highest ideal value on attribute k. Given any two agents i, j with preferred alternatives x^i and x^j, let $x_k^{h(i,j)} \equiv \max\{x_k^i, x_k^j\}$ and let $x_k^{l(i,j)} \equiv \min\{x_k^i, x_k^j\}$.

I find the necessary and sufficient conditions to represent the preferences of every agent by utility functions that are linear in generalized weighted city block distances in some space that is common to all agents.

Axiom 5 *(Linear Representability) For any $i, j \in N$, $\forall k \in A$, $\forall x_k^a, x_k^b, x_k^c \in X_k$ such that $x_k^a \leq_k x_k^{l(i,j)} \leq_k x_k^b \leq_k x_k^{h(i,j)} \leq_k x_k^c$, $\forall x_{-k} \in X_{-k}$ and $\forall \alpha \in [0,1]$, given $p_k^a, p_k^b, p_k^c \in \Delta X_k$ such that $p_k^a(x_k^{\min}) = \alpha$, $p_k^a(x_k^{l(i,j)}) = 1-\alpha$, $p_k^b(x_k^{h(i,j)}) = \alpha$, $p_k^b(x_k^{l(i,j)}) = 1-\alpha$, $p_k^c(x_k^{\max}) = \alpha$ and $p_k^c(x_k^{h(i,j)}) = 1-\alpha$,*

$$(p_k^z; x_{-k}) \sim_i (x_k^z, x_{-k}) \Longleftrightarrow (p_k^z; x_{-k}) \sim_j (x_k^z, x_{-k}) \text{ for any } z \in \{a, b, c\}.$$

Linear representability has a very simple interpretation. Fixing the value of all attributes except k, and evaluating lotteries that assign different values to attribute k, if agent l_k and agent i agree in their ordinal preference among all the possible outcomes of the lotteries, then they agree on their ranking of the lotteries as well. Agents l_k and i share the same ranking among all lotteries on attribute k that assign positive probability only to values that are no greater than the ideal value of l_k. Similarly, for lotteries that are in any event above x_k^i, agents agree that they want less of attribute k, and they agree on their ranking of these lotteries. In the intermediate interval between their two ideal policies, the agents have opposite rankings over sure outcomes: Agent l_k wants less, agent i wants more. Linear representability requires that if agent l_k is indifferent between a lottery in this interval and a sure outcome, agent i must be indifferent as well. An intuition is that in this region the agents are in a zero-sum game: Whatever l_k gains, i loses, so if l_k is indifferent between two lotteries, i must be indifferent as well.

Linear representability, together with separability, multidimensional single peakedness and the standard expected utility conditions assumed throughout this section, guarantees that it is possible to construct a spatial representation $f(X)$ such that we can represent the preferences of every $i \in N$ by a function that is linearly decreasing in the generalized weighted city block distance to the ideal point of the agent.

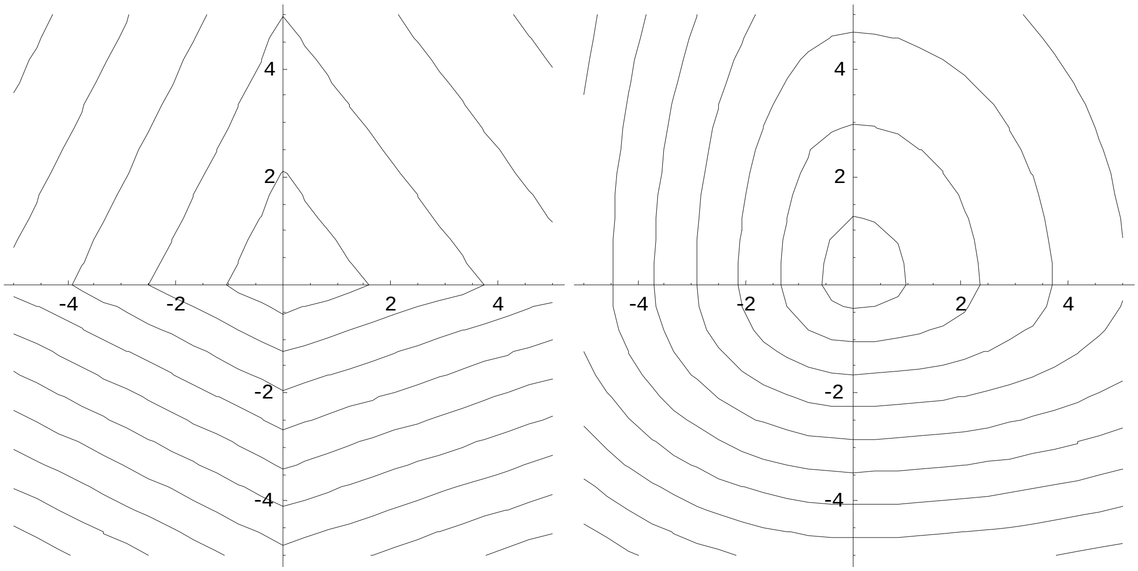
Proposition 3 *A common spatial representation $f \in \mathcal{F}$ such that for each $i \in N$ the utility function $u_i(x)$ is linearly decreasing in the generalized weighted city block distance to $f(x^i)$ for some vector of weights $\lambda \in \mathbb{R}_+^{2K}$ exists if and only if preferences $\succsim_N$ satisfy linear representability.*

Succinctly, and a bit informally, if agents agree on lotteries when they agree on sure outcomes and they have exactly opposite preferences over lotteries when they have exactly opposite preferences over sure outcomes, then their ordinal preferences over multi-attribute objects can be represented in a common space such that these preferences can all be represented by utility functions that are linearly decreasing in a generalized weighted city block distance to the respective ideal points.

Proposition 3 has very important consequences in political competition over policy bundles with multiple policy dimensions: If agents have weighted city block preferences, for an open set of distributions of weights, there exist policy bundles that are in the majority voting rule core, so they cannot be defeated by any other policy.[5]

With regards to representability by the Euclidean distance, Bogomolnaia and Laslier [4] show that any profile of preference relations can be represented by Euclidean preferences in a space with a sufficiently large number of dimensions. With a fixed number of dimensions, the goal of representing preferences by means of a Euclidean

[5] See Rae and Taylor [29], Wendell and Thorson [36], McKelvey and Wendell [23] and Humphreys and Laver [14].



Generalized weighted city block (δ=1) Generalized weighted Euclidean (δ=2)

Figure 2: More general classes of indifference curves.

utility function in a common space becomes a much more difficult task, and the conditions on preferences become very restrictive. For any vector of weights $w \in \mathbb{R}_+^K$, the utility function $u_i(y) = -\sum_{k=1}^{K} w_k(y_k - y_k^i)^2$, is a generalization of quadratic Euclidean preferences that assigns different weights to each dimension. In $\mathbb{R}^2$, indifference curves become ellipses instead of circles. I consider a further generalization that makes weights $w_k(y_k, y_k^i)$ a function of $y_k - y_k^i$, allowing for different weights in each direction within each dimension. Allowing different weights in each direction in only one dimension turns the ellipses in $\mathbb{R}^2$ into egg shaped ovoids. Allowing different weights in each direction in every dimension breaks every symmetry, generating shapes like those depicted in the second graph in figure 2. Upper contour sets are convex and smooth.

Definition 2 *Given any vector of weights* $w \equiv (w_{1+}, w_{1-}, w_{2+}, w_{2-}, ..., w_{K+}, w_{K-}) \in \mathbb{R}_+^{2K}$, *a generalized weighted Euclidean distance with weights* w *is a function* $g : \mathbb{R}^K \times$

$\mathbb{R}^K \longrightarrow \mathbb{R}_+$ *such that for any* $y, y' \in \mathbb{R}^K$,

$$g(y, y') = \left(\sum_{k=1}^{K} [w_k(y_k, y'_k)](y_k - y'_k)^2 \right)^{1/2},$$
$$\textit{where } w_k(y_k, y'_k) = \{w_{k+} \textit{ if } y_k \geq y'_k,\ w_{k-} \textit{ if } y_k < y'_k\}\ \forall k \in A.$$

I state a result parallel to proposition 3, finding a condition on preferences so that utility functions are quadratic on the Euclidean distance in each dimension. This condition, unfortunately, is more complex and cumbersome than the linear city block case, and its interpretation is not as intuitive.

Given an arbitrary attribute k, fix the values on all other attributes at x_{-k}. Recall that $x_k^{l_k}$ is the lowest ideal value of any agent on attribute k. Unlike under linear representability, in order to be able to represent preferences over X_k by a quadratic function, if $x_k^i \neq x_k^{l_k}$, then it must be that i and l_k disagree on their evaluation of lotteries over X_k even in the region of X_k where l_k and i agree, that is, agents l_k and i must disagree on their preference relation of lotteries with support below $x_k^{l_k}$ and lotteries with support over x_k^i, and they must disagree in a very specific way. Fix $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$. For every $x_k \geq_k x_k^{l_k}$, let $f_k(x_k)$ be such that the utility function of l_k defined over x_k given x_{-k} is quadratic from $f_k(x_k^{l_k})$ to $f_k(x_k^{\max})$. Let $\gamma^i \equiv f_k(x_k^i)$, so γ^i is the distance between the ideal points of l_k and i as fixed according to the quadratic utility representation of the preferences of l_k. Then, if $\gamma = 0$, the preferences of l_k and i must coincide for every lottery on attribute k with support above $x_k^{l_k}$. However, as $\gamma \longrightarrow 1$, agent i must be increasingly more risk averse than l_k over any lottery on X_k with support above x_k^i. In particular, let $\widehat{x}_k$ the midpoint between x_k^i and $x_k^{\max}$ according to l_k so $f_k(\widehat{x}_k) = \frac{1+\gamma^i}{2}$. While for any γ^i, agent i is indifferent between $\widehat{x}_k$ for sure and a lottery that assigns probability 0.25 to $x_k^{\max}$ and probability 0.75 to x_k^i, if l_k is indifferent between $\widehat{x}_k$ for sure and a lottery that assigns probability α to $x_k^{\max}$ and probability $1-\alpha$ to x_k^i, then α is an increasing function of γ^i, starting at $\alpha = 0.25$ for $\gamma^i = 0$ and converging to $\alpha = 0.5$ as γ^i converges to 1, so that l_k must be less risk averse than i, or i relatively more risk averse than l_k, the further away that l_k locates $f_k(x_k^i)$. The full condition is as follows.

Axiom 6 *(Quadratic representability) For any $k \in A$, $\forall x_{-k} \in X_{-k}$ and $\forall i \in N$, let $\gamma_i \in [0,1]$ and $p \in \Delta X$ with $p(x_k^{\max}, x_{-k}) = (\gamma_i)^2$ and $p(x_k^{l_k}, x_{-k}) = 1 - (\gamma_i)^2$ be such that $p \sim_{l_k} (x_k^i, x_{-k})$, and let $\gamma_0 \in \mathbb{R}$ and $q \in \Delta X$ with $q(x_k^{\min}, x_{-k}) = \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^2$ and $q(x_k^{h_k}, x_{-k}) = 1 - \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^2$ be such that $q \sim_{h_k} (x_k^{l_k}, x_{-k})$. Then, $\forall x_{-k} \in X_{-k}$ and $\forall i \in N$:*

- *For any $x_k^a \geq_k x_k^i$ and $\forall p, q \in \Delta X$ such that $p(x_k^{\max}, x_{-k}) = \alpha_i, p(x_k^i, x_{-k}) = 1 - \alpha_i$, $q(x_k^{\max}, x_{-k}) = \alpha_{l_k}, q(x_k^i, x_{-k}) = 1 - \alpha_{l_k}$, and $p \sim_i (x_k^a, x_{-k})$,*

$$q \sim_{l_k} (x_k^a, x_{-k}) \Longleftrightarrow \alpha_{l_k} = \frac{\alpha_i + \gamma_i(2\sqrt{\alpha_i} - \alpha_i)}{1 + \gamma_i}.$$

- *For any $x_k^b \in X_k$ such that $x_k^{l_k} \leq_k x_k^b \leq_k x_k^i$, $\forall p, q \in \Delta X$ such that $p(x_k^i, x_{-k}) = \alpha_i, p(x_k^{l_k}, x_{-k}) = 1-\alpha_i$, $q(x_k^{l_k}, x_{-k}) = \alpha_{l_k}, q(x_k^i, x_{-k}) = 1-\alpha_{l_k}$, and $p \sim_i (x_k^b, x_{-k})$,*

$$q \sim_{l_k} (x_k^b, x_{-k}) \Longleftrightarrow \alpha_{l_k} = 2\sqrt{1 - \alpha_i} - (1 - \alpha_i).$$

- *For any $x_k^c \leq_k x_k^i$ and $\forall p, q \in \Delta X$ such that $p(x_k^{\min}, x_{-k}) = \alpha_i, p(x_k^i, x_{-k}) = 1 - \alpha_i$, $q(x_k^{\min}, x_{-k}) = \alpha_{h_k}, q(x_k^i, x_{-k}) = 1 - \alpha_{h_k}$, and $p \sim_i (x_k^c, x_{-k})$,*

$$q \sim_{h_k} (x_k^c, x_{-k}) \Longleftrightarrow \alpha_{h_k} = \frac{\alpha_i(\gamma_i + \gamma_0) + 2\sqrt{\alpha_i}(\gamma_{h_k} - \gamma_i)}{2\gamma_{h_k} + \gamma_0 - \gamma_i}.$$

With this condition, I construct a mapping $f(X)$ such that $f_k(x_k) : X_k \longrightarrow [-\gamma_0, 1]$, where on attribute k, the lowest ideal point is mapped to 0, the ideal point of any other agent i is at γ^i, and preferences over points in this map are generalized weighted quadratic Euclidean.

Proposition 4 *A spatial representation* $f \in \mathcal{F}$ *and a vector of weights* $w^i = (w^i_{1-}, w^i_{1+}, ..., w^i_{K-}, w^i_{K+}) \in \mathbb{R}^{2K}_+$ *for each* $i \in N$ *such that the utility function* $u_i(x)$ *is quadratic decreasing in the generalized weighted Euclidean distance from* $f(x)$ *to the ideal point* $f(x^i)$ *with weights* w^i *exists if and only if preferences* $\succsim_N$ *satisfy quadratic representability.*

While quadratic representability may seem unduly restrictive, note that it characterizes the set of preferences representable by a family of utility functions that is much more general than the quadratic utility functions routinely used in multidimensional political economy models. Preferences representable by a quadratic Euclidean utility function must satisfy quadratic representability, very stringent symmetry conditions in each dimension (the formal conditions are available from the author), and the altogether implausible condition that all agents assign the same relative importance to each attribute.

It is helpful to compare quadratic representability to linear representability. Linear representability can be rewritten in terms that follow the structure of quadratic representability, and then, linear representability holds if $q^z \sim_{l_k} (x^z_k, x_{-k}) \Longleftrightarrow \alpha_{l_k} = \alpha_i$ for the first and third case $z = a$ and $z = c$, while $q^b \sim_{l_k} (x^b_k, x_{-k}) \Longleftrightarrow \alpha_{l_k} = 1 - \alpha_i$ for the second case with $x^{l_k}_k \leq_k x^b_k \leq_k x^i_k$. This formulation is less compact, but it perhaps makes it clearer that agents must agree on lotteries if they agree on sure outcomes (first and third case), and they must have opposite preferences over lotteries when they have opposite preferences over sure outcomes.

For the purpose of a clearer intuition, choose an arbitrary attribute k, fix the values in all other attributes at x_{-k}, assume that there is a continuum of values in X_k that can be indexed by real numbers so that $X_k = [0, 1]$, assume that preferences are continuous in X_k, and let l_k be denoted simply by l. Then for any $i \in N$, we can find a value x^m_k between the ideal values of agents l and i such that each l and i are indifferent between x^m_k and a lottery on attribute k that grants $j \in \{l, i\}$ her ideal value x^j_k with probability α and the ideal point of the other player with probability

$1-\alpha$. Formally, $p(x_k^l)=\alpha$, $p(x_k^i)=1-\alpha$, $q(x_k^i)=\alpha$ and $q(x_k^l)=1-\alpha$, then $(x_k^m, x_{-k}) \sim_l p$ and $(x_k^m, x_{-k}) \sim_i q$. A representation $f_k(x_k)$ that maps x_k^m to the midpoint between $f_k(x_k^l)$ and $f_k(x_k^i)$ generates the same risk attitude for agents i and j over lotteries p and q. Linear representability requires that $\alpha = 0.5$, which I interpret as risk neutrality, while quadratic representability requires $\alpha = 0.75$, which I interpret as risk aversion.

As I argue in the introduction, individual risk attitudes given an endogenous spatial representation are not meaningful: It is always possible to make the utility function of an individual agent risk loving or risk averse on any given dimension by changing the spatial representation of alternatives. However, if we construct a spatial representation that makes every agent have the same shape of the utility functions, then we can interpret these shapes as a collective degree of risk aversion in society. Given agents i and j and an intermediate value x_k^m such that $x_k^i \leq_k x_k^m \leq_k x_k^j$ and both i regard x_k^m equally in terms of how good it is relative to their respective ideal values, then in order to give the two agents the same utility representation, we must locate $f_k(x_k^m)$ as exactly the midpoint between $f_k(x_k^i)$ and $f_k(x_k^j)$. I illustrate this in figure 3. Given an arbitrary attribute x, an arbitrary x_{-k}, and a utility function for each agent that is scaled so that $u_i(x_k^i, x_{-k}) - u_i(x_k^j, x_{-k}) = u_j(x_k^j, x_{-k}) - u_j(x_k^i, x_{-k})$, the left panels depict these utility functions given an arbitrary representation of x_k^m, which is the point at which the two utility functions cross. The right panels represent the shape of the utility functions once we remap x_k^m to the midpoint between x_k^i and x_k^j. The top two panels depict a case in which the value at which the two utility functions cross is a half of $u_i(x_k^i, x_{-k})-$ $u_i(x_k^j, x_{-k})$, and in this case, once we choose the appropriate spatial representation $f_k(x_k^m) = \frac{f_k(x_k^i)+f_k(x_k^j)}{2}$ then the utility functions of both agents are linear. In the bottom two panels, the value at which the utility functions cross is $0.75[u_i(x_k^i, x_{-k})-$ $u_i(x_k^j, x_{-k})]$ and once we locate x_k^m as the midpoint in the bottom right panel, the utility representation is quadratic in distance (the graph at the bottom right is an approximation to a quadratic function with only three points,

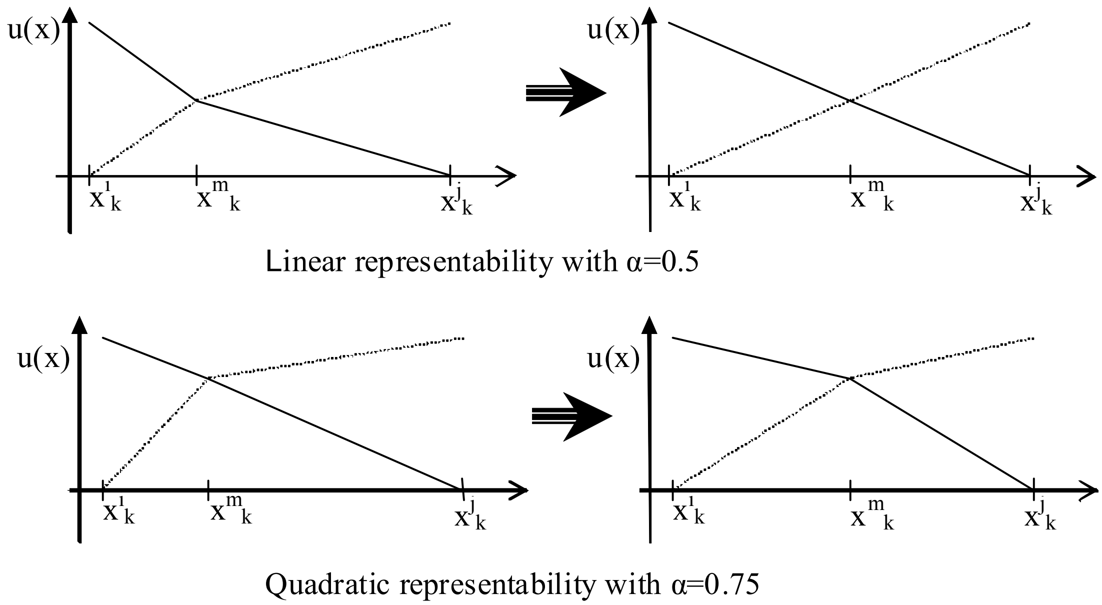


Figure 3: A representation $f(X)$ such that i and j have the same risk attitude.

adding more points would generate the expected curvature in the figure). The value of α that makes agents indifferent between x_m^k for sure or a lottery between their ideal or the other agent's ideal value then has an interpretation in terms of risk preference: It identifies the only risk attitude that is consistent with both agents having the same risk attitude. It would always be possible to make one agent risk averse and the other risk loving by letting $f_k(x_k^m)$ be very close to $f_k(x_k^i)$ or $f_k(x_k^j)$, but if $\alpha > 0.5$, then it would never be possible to make both i and j risk loving, whereas if $\alpha < 0.5$, it would not be possible to make both i and j risk averse. Hence, I interpret the value α as a measure of the average or aggregate degree of risk aversion in society.

Whether preferences over multi-attribute alternatives are such that $\alpha = 0.5$ or whether $\alpha = 0.75$ is a testable empirical question. Evidence that $\alpha \approx 0.75$ would support the assumption of quadratic Euclidean preferences. On the other hand, evidence that $\alpha \approx 0.5$ would suggest that, albeit standard, the assumption of quadratic Euclidean preferences is unwarranted and it is appropriate to assume instead linear city block preferences, with positive implications for existence of majority core outcomes in multidimensional policy competition. More generally, each value α corresponds to a particular shape of the indifference curves, and to a utility function that is linearly decreasing in the δ power of the weighted distance $d^{\delta}(f(x), f(x^i))$ that defines the indifference curves.

Proposition 5 *Let there exist $\delta > 0$, a spatial representation $f \in \mathcal{F}$ and a vector of weights $w^i \in \mathbb{R}_+^{2K}$ for each $i \in N$ such that $u_i(x) = -\sum_{k=1}^{K} w_k(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^{\delta}$, where $w_k(x_k, x_k^i) = \{w_{k-}$ if $x_k \leq_k x_k^i$, w_{k+} if $x_k \geq_k x_k^i\}$. If there exist $\{k \in A,\ i \in N$, with $x_k^i \neq x_k^{l_k}$, $\alpha \in (0,1)$, $x_{-k} \in X_{-k}$ and $x_k^m \in X_k\}$ such that $\{(x_k^m, x_{-k}) \sim_{l_k} p$ and $(x_k^m, x_{-k}) \sim_i q$ given $p(x_k^{l_k}, x_{-k}) = \alpha$, $p(x_k^i, x_{-k}) = 1 - \alpha$, $q^i(x_k^i, x_{-k}) = \alpha$ and $q^i(x_k^{l_k}, x_{-k}) = 1 - \alpha\}$, then*

$$\delta = \frac{-\ln(1-\alpha)}{\ln 2}.$$

Note how if $\alpha = 0.5$, then $\delta = \frac{-\ln(1-0.5)}{\ln 2} = \frac{-\ln(1/2)}{\ln 2} = 1$, the case illustrated at the top of figure 3; whereas, if $\alpha = 0.75$, then $\delta = \frac{-\ln(1/4)}{\ln 2} = \frac{\ln 4}{\ln 2} = 2$, the case illustrated at the bottom of figure 3. Once we empirically estimate the value $\widehat{\alpha}$, we know what the appropriate utility representation of the preferences of the agents within the family of functions $u_i(x) = -\sum_{k=1}^{K} w_k(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^{\delta}$. If $\widehat{\alpha} \leq 0.5$, then the appropriate $\delta(\widehat{\alpha}) \leq 1$ and the implications are similar to city block case with $\delta = 1$: The core of majority rule is not necessarily empty. If, on the other hand, $\widehat{\alpha} > 0.5$ and the appropriate $\delta(\widehat{\alpha}) > 1$, then the utility functions are smooth and the core remains generically empty. Choosing the utility function with the proper parameter δ may still significantly improve the results of empirical models that currently fix $\delta = 2$ without support for this assumption, such as, for instance, the ideal point estimation model by Clinton, Jackman and Rivers [6].

Proposition 5 is simple. At the same time, it is the consequence of a less transparent, but more powerful result. It is possible to find analogous conditions to linear representability and quadratic representability for any $\delta > 0$, so that the preferences of every agent are representable by a utility function that is linearly decreasing in the

δ power of a generalized weighted distance function $d^{\delta}(y, y^{i})$. I state this as a theorem. Propositions 3 and 4 are the particular cases for $\delta = 1$ and $\delta = 2$, and while they are attractive for their greater simplicity, they follow as corollaries from the more general theorem 6.

Theorem 6 *For any $\delta \in \mathbb{R}_{++}$, a spatial representation $f \in \mathcal{F}$ and a vector of weights $w^{i} = (w^{i}_{1-}, w^{i}_{1+}, ..., w^{i}_{K-}, w^{i}_{K+}) \in \mathbb{R}^{2K}_{+}$ for each $i \in N$ such that $u_i(x) = -\sum_{k=1}^{K} w_k(x_k, x^{i}_{k})|f_k(x_k) - f_k(x^{i}_{k})|^{\delta}$, where $w_k(x_k, x^{i}_{k}) = \{w_{k-}$ if $x_k \leq_k x^{i}_{k}$, w_{k+} if $x_k \geq_k x^{i}_{k}\}$ exists if and only if the following condition holds.*

For any $k \in A$, $\forall x_{-k} \in X_{-k}$, and $\forall i \in N$, let $\gamma_i \in [0,1]$ and $p \in \Delta X$ with $p(x^{\max}_{k}, x_{-k}) = (\gamma_i)^{\delta}$ and $p(x^{l_k}_{k}, x_{-k}) = 1 - (\gamma_i)^{\delta}$ be such that $p \sim_{l_k} (x^{i}_{k}, x_{-k})$, and let $\gamma_0 \in \mathbb{R}$ and $q \in \Delta X$ with $q(x^{\min}_{k}, x_{-k}) = \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^{\delta}$ and $q(x^{h_k}_{k}, x_{-k}) = 1 - \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^{\delta}$ be such that $q \sim_{h_k} (x^{l_k}_{k}, x_{-k})$. Then, $\forall x_{-k} \in X_{-k}$ and $\forall i \in N$:

i) For any $x^{a}_{k} \geq_k x^{i}_{k}$ and $\forall p, q \in \Delta X$ such that $p(x^{\max}_{k}, x_{-k}) = \alpha_i, p(x^{i}_{k}, x_{-k}) = 1 - \alpha_i$, $q(x^{\max}_{k}, x_{-k}) = \alpha_{l_k}, q(x^{i}_{k}, x_{-k}) = 1 - \alpha_{l_k}$, and $p \sim_i (x^{a}_{k}, x_{-k})$,

$$q \sim_{l_k} (x^{a}_{k}, x_{-k}) \Longleftrightarrow \alpha_{l_k} = \frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^{\delta} - \gamma_i^{\delta}}{1 - \gamma_i^{\delta}}.$$

ii) For any $x^{b}_{k} \in X_k$ such that $x^{l_k}_{k} \leq_k x^{b}_{k} \leq_k x^{i}_{k}$, $\forall p, q \in \Delta X$ such that $p(x^{i}_{k}, x_{-k}) = \alpha_i, p(x^{l_k}_{k}, x_{-k}) = 1-\alpha_i$, $q(x^{l_k}_{k}, x_{-k}) = \alpha_{l_k}, q(x^{i}_{k}, x_{-k}) = 1-\alpha_{l_k}$, and $p \sim_i (x^{b}_{k}, x_{-k})$,

$$q \sim_{l_k} (x^{b}_{k}, x_{-k}) \Longleftrightarrow \alpha_{l_k} = 1 - \left(1 - (1-\alpha_i)^{1/\delta}\right)^{\delta}.$$

iii) For any $x^{c}_{k} \leq_k x^{i}_{k}$ and $\forall p, q \in \Delta X$ such that $p(x^{\min}_{k}, x_{-k}) = \alpha_i, p(x^{i}_{k}, x_{-k}) = 1 - \alpha_i$, $q(x^{\min}_{k}, x_{-k}) = \alpha_{h_k}, q(x^{i}_{k}, x_{-k}) = 1 - \alpha_{h_k}$, and $p \sim_i (x^{c}_{k}, x_{-k})$,

$$q \sim_{h_k} (x^{c}_{k}, x_{-k}) \Longleftrightarrow \alpha_{h_k} = \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^{\delta} - (\gamma_h - \gamma_i)^{\delta}}{(\gamma_0 + \gamma_h)^{\delta} - (\gamma_h - \gamma_i)^{\delta}}.$$

Note how the three conditions in theorem 6 reduce to the much simpler linear representability condition if $\delta = 1$, and to quadratic representability if $\delta = 2$.

At the same time, it is straightforward to generalize theorem 6 further, letting the preferences of the agents be such that we can find a common spatial representation $f(X)$ where the utility of each agent i is linearly decreasing in the δ_i power of a generalized weighted version of the distance function $d^{\delta_i}(f(x), f(x^i))$. The expansion of the class of utility functions allows for each agent to have her own degree of curvature in her indifference curves; for instance, agent i may have linear city block preferences while agent j has quadratic Euclidean preferences in the chosen spatial representation. For any vector $\delta = (\delta_1, ..., \delta_n) \in \mathbb{R}^n_{++}$, the statement and proof of the more general theorem convey limited new intuition, as they merely substitute δ for the appropriate δ_i at each step. This is omitted, but available from the author. Note that indifference curves are smooth if and only if $\delta_i > 1$, so Plott's theorem [27] on the emptiness of the core, or McKelvey's [21] theorem on the intransitivity of majority rule break down if $\delta_i \leq 1$ for even a small subset of agents.

3 Discussion

Given alternatives that are objects with multiple attributes, it is tempting to represent these objects as points in a vector space. However, unless the values within each attribute are objectively quantifiable, any spatial representation is subjective, arbitrary, or made for convenience. The primitive space of objects is not a subset of a vector space. The primitive is merely a subset of the Cartesian product of the set of possible values in each attribute. Any assumption on preferences over alternatives in the spatial representation of the space of objects is not an assumption on primitives, such as preferences over alternatives; it is a joint assumption on preferences over alternatives, and on the chosen spatial representation of the preferences.

The theoretical contribution of this paper is to find simple and intuitive necessary and sufficient conditions on the preference relation over the primitive set of alternatives, such that for any $\delta > 0$, there exists a spatial representation of these alternatives under which preferences can be represented by a utility function that is decreasing in the δ power of the distance d^δ to an ideal point in the space, where the d^δ belongs to a large class of simple functions, of which Minkowski's [25] family of metric functions is a small subset.

I show that choosing the appropriate spatial representation $f(X)$, the preferences of a single agent can be represented by a utility function $u(x) = -[d^\delta(x,0)]^\delta = -\sum_{k=1}^{K} f_k(x_k)^\delta$ for any $\delta > 0$ if and only if the preferences are separable across attributes, and they are single peaked within each attribute.

In a society with multiple agents, additional conditions on preferences guarantee that there exists a spatial representation $f(X)$ common to all agents such that the utility of every agent is linearly decreasing in the δ power of a weighted version of the distance function $d^\delta(f(x), f(x^i))$. These additional conditions are simpler and more intuitive for city block utility functions than for smooth utility functions such as quadratic Euclidean.

An implication of the results in this paper that some received wisdom that relies on the Euclidean distance and more generally on smooth indifference curves perhaps should be reevaluated. I have characterized the set of preferences that are representable by utility functions that belong to a family of functions indexed by a parameter δ. If preferences are representable by a utility function in this family with $\delta > 1$, then the results on smooth utility functions apply, but if preferences are representable by a utility function with $\delta \leq 1$, then these results, including the emptiness of the core and the global intransitivities of majority rule,[6] do not apply. I have suggested an empirical test to estimate δ in laboratory experiments. The next step in the research agenda is to conduct this test and find out if there is any evidence in support of the standard assumption of Euclidean or at least smooth utility functions.

[6] See Plott [27], McKelvey and Schofield [22] and McKelvey [20] and [21].

Appendix

Proof of theorem 1.

Proof. ($\Longrightarrow$) Suppose preference $\succsim$ is not multi-attribute single peaked. Let x_k^* be the ideal policy bundle of the agent. Since preferences are not multi-attribute single peaked, there exists $k \in \{1, ..., K\}$ and $x^1 = (x_k^1, x_{-k}^*)$, $x^2 = (x_k^2, x_{-k}^*)$ such that either $x_k^1 \leq x_k^2 \leq x_k^*$ or $x_k^* \leq x_k^2 \leq x_k^1$, and $x^1 \succ x^2$ so $u((x_k^1, x_{-k}^*)) > u((x_k^2, x_{-k}^*))$. Note that for any f,

$$\sum_{h=1}^{K} |f_h(x_h^1)| - \sum_{h=1}^{K} |f_h(x_h^2)| = |f_k(x_k^1)| - |f_k(x_k^2)|$$

and

$$\sum_{h=1}^{K} |f_h(x_h^*)| - \sum_{h=1}^{K} |f_h(x_h^2)| = |f_k(x_k^*)| - |f_k(x_k^2)|.$$

Since f_k is strictly increasing in $\geq_k$, $\min\{f_k(x_k^*), f_k(x_k^1)\} < f_k(x_k^2) < \max\{f_k(x_k^*), f_k(x_k^1)\}$. But $u(x^2) < \min\{u(x^1), u(x^*)\}$. Hence $u(x)$ is not decreasing in $\sum_{h=1}^{K} |f_h(x_h)|$. Suppose $u(x) = -\sum_{k=1}^{K} \left|f_k^\delta(x_k)\right|^\delta$ for some $\delta > 1$ and some f^δ. For each k, let $g_k(x_k)$ be such that $|g_k(x_k)| = \left|f_k^\delta(x_k)\right|^\delta$ and $g_k(x_k) \geq 0 \Longleftrightarrow f_k^\delta(x_k) \geq 0$. Note that under the spatial representation $g = (g_1, ..., g_K)$, $u(x) = -\sum_{k=1}^{K} |g_k(x_k)|$. But $u(x)$ is not decreasing in $\sum_{h=1}^{K} |f_h(x_h)|$ for any f, hence this is a contradiction.

Suppose $\succsim$ is not modular. Then $\exists x^1, x^2 \in X$ s.t. $L(x^1, x^2) \nsim L(x^1 \vee x^2, x^1 \wedge x^2)$, which implies $u(x^1) + u(x^2) \neq u(x^1 \vee x^2) + u(x^1 \wedge x^2)$. However, if we let $x^j \equiv x^1 \vee x^2$ and $x^m \equiv x^1 \wedge y^1$,

$$\sum_{k=1}^{K} |f_h(x_h^1)| + \sum_{k=1}^{K} |f_h(x_h^2)| = \sum_{k=1}^{K} |f_h(x_h^j)| + \sum_{k=1}^{K} |f_h(x_h^m)|.$$

So $u(x) \neq -\sum_{k=1}^{K} |f_k(x_k)|$. The same transformation as in the previous paragraph extends this proof to any δ.

($\Longleftarrow$) Suppose $\succsim$ is modular and multi-attribute single peaked. Let (x_k^*, x_{-k}^*)

denote x^*. For each $k \in \{1, 2, ..., K\}$, construct $f_k^1 : X_k \longrightarrow \mathbb{R}$ as follows: $f_k^1(x_k^*) = 0$; for any $x_k^j \in X_k$ such that $x_k^j \leq_k x_k^*$, $f_k^1(x_k^j) = u(x_k^j, x_{-k}^*) - u(x_k^*, x_{-k}^*)$; and for any $x_k^j \in X_k$ such that $x_k^j \geq_k x_k^*$, $f_k^1(x_k^j) = u(x_k^*, x_{-k}^*) - u(x_k^j, x_{-k}^*)$. By multi-attribute single peakedness, these functions f_k^1 are strictly increasing in $\geq_k$. By construction, $u(x)$ is linearly decreasing in $\sum_{k=1}^{K} |f_k^1(x_k)|$ for any x such that $x_h = x_h^*$ for any $h \neq k$.

I complete the proof by induction. Suppose that $u(x)$ is linearly decreasing in $\sum_{k=1}^{K} |f_k^1(x_k)|$ for any x such that $x_h = x_h^*$ for any $h \in A_m$, where $A_m \subset A$. I want to show that $u(z)$ is also linearly decreasing in $\sum_{k=1}^{K} |f_k^1(x_k)|$ for any $z \in X$ such that $z_h = z_h^*$ for any $h \in A_{m-1}$, where $A_{m-1} \subset A_m$ and $|A_{m-1}| = |A_m| - 1$. Let i be a dimension such that $z_i = x_i \neq x_i^*$. Let j be the dimension such that $z_j \neq x_j = x_j^*$. Let $w \in X$ be such that $w_i = x_i^*; w_j = x_j^*$ and $w_k = z_k$ for $k \notin \{i, j\}$. Let $y \in X$ be such that $y_i = x_i^*$ and $y_k = z_k$ for $k \neq i$. Then $\{w, y, x, z\}$ is a lattice, where $w \succsim x, y \succsim z$. By the inductive hypothesis, $u(w)$ is linearly decreasing in $\sum_{k=1}^{K} |f_k^1(w_k)|$.

$$u(w) = -\sum_{k=1}^{K} |f_k^1(w_k)| = -\sum_{k \notin \{i,j\}} |f_k^1(w_k)|$$

where the second equality follows from $w_k = x_k^*$ for $k \in \{i, j\}$. Since, again by the inductive hypothesis, $u(x)$ and $u(y)$ are also linearly decreasing in $\sum_{k=1}^{K} |f_k^1(x_k)|$ and $\sum_{k=1}^{K} |f_k^1(y_k)|$,

$$u(x) = u(w) - |f_i^1(x_i) - f_i^1(x_i^*)|$$

and

$$u(y) = u(w) - |f_j^1(y_j) - f_j^1(x_j^*)|.$$

By modularity,

$$\begin{aligned}
u(z) &= u(y)+u(x)-u(w) = u(w) - |f_i^1(x_i) - f_i^1(x_i^*)| - |f_j^1(y_j) - f_j^1(x_j^*)| \\
&= u(w) - |f_i^1(z_i) - f_i^1(x_i^*)| - |f_j^1(z_j) - f_j^1(x_j^*)| \\
&= -\sum_{k\notin\{i,j\}} |f_k^1(w_k)| - \sum_{k\in\{i,j\}} |f_k^1(z_k)| \\
&= -\sum_{k=1}^{K} |f_k^1(z_k)|.
\end{aligned}$$

Since the inductive hypothesis is true, as shown, for $|A_m| \geq K-1$, it is true by the inductive argument for any size of A_m, and therefore, for any $x \in X$, $u(x)$ is linearly decreasing in distance to $\sum_{k=1}^{K} |f_k^1(x_k)|$. For any $\delta > 0$, let f_k^δ be such that $\left|f_k^\delta(x_k)\right| = |f_k^1(x_k)|^{1/\delta}$ and $f_k^\delta(x_k) \geq 0 \iff f_k^1(x_k) \geq 0$. Then

$$u(x) = -\sum_{k=1}^{K} \left|f_k^1(x_k)\right| = -\sum_{k=1}^{K} \left|f_k^\delta(x_k)\right|^\delta.$$

■

Proof of claim 2

Proof. Since l is strictly increasing, the shape of the indifference curves is given by $\left(\sum_{k=1}^{K} |f_k(x_k)|^\delta\right)^{1/\delta}$, so preferences are multi-attribute single peaked. Theorem 1 then applies and preferences are representable by the expected utility of $v(x) = -\sum_{k=1}^{K} |f_k^\delta(x_k)|^\delta$. By Von Neumann and Morgenstern's [35] expected utility theorem, preferences are representable by the expected utility of another function $u(x)$ if and only if $u(x)$ is an affine transformation of $v(x)$, which requires $l(d) = \alpha + \beta d^\delta$. ■

Proof of proposition 3

Proof. By contradiction. Let $d_{\lambda^i}(y, y^i)$ denote the generalized weighted city block distance with weights $\lambda^i \in \mathbb{R}^{2K}$. Suppose that $u_i(x) = -d_{\lambda^i}(f(x), f(x^i))$ for every $i \in N$, but linear representability fails for the case $z = a$ in the statement of linear representability. Then, for any $k \in A$ and $\forall x_{-k} \in X_{-k}$, $\exists\{i, j \in N$, $\alpha \in [0,1]$ and

$x_k^a \leq_k x_k^{l(i,j)}\}$ such that given $p_k^a \in \Delta X_k$ with $p_k^a(x_k^{\min}) = \alpha$ and $p_k^a(x_k^{l(i,j)}) = 1 - \alpha$, $(p_k^a; x_{-k}) \sim_i (x_k^a, x_{-k})$ and $(p_k^a; x_{-k}) \nsim_j (x_k^a, x_{-k})$. Simplify notation to let $l = l(i, j)$. In utility terms, $(p_k^a; x_{-k}) \sim_i (x_k^a, x_{-k})$ implies

$$\alpha u_i((x_k^{\min}, x_{-k})) + (1 - \alpha) u_i((x_k^l, x_{-k})) = u_i((x_k^a, x_{-k})),$$

which, since $u_i(x)$ is linearly decreasing in

$$d_{\lambda^i}(f(x), f(x^i)) = \lambda_k^i |f_k(x_k) - f_k(x_k^i)| + \sum_{m \neq k} \lambda_m^i |f_m(x_m) - f_m(x_m^i)|,$$

implies

$$\begin{aligned}
\alpha |f_k(x_k^{\min}) - f_k(x_k^i)| + (1 - \alpha)|f_k(x_k^l) - f_k(x_k^i)| &= |f_k(x_k^a) - f_k(x_k^i)| \\
\alpha f_k(x_k^i) - \alpha f_k(x_k^{\min}) + (1 - \alpha) f_k(x_k^i) - (1 - \alpha) f_k(x_k^l) &= f_k(x_k^i) - f_k(x_k^a) \\
\alpha (f_k(x_k^l) - f_k(x_k^{\min})) &= f_k(x_k^l) - f_k(x_k^a).
\end{aligned}$$

In utility terms, $(p_k^a; x_{-k}) \nsim_j (x_k^a, x_{-k})$ implies

$$\begin{aligned}
\alpha |f_k(x_k^{\min}) - f_k(x_k^h)| + (1 - \alpha)|f_k(x_k^i) - f_k(x_k^h)| &\neq |f_k(x_k^a) - f_k(x_k^h)| \\
\alpha f_k(x_k^h) - \alpha f_k(x_k^{\min}) + (1 - \alpha) f_k(x_k^h) - (1 - \alpha) f_k(x_k^l) &\neq f_k(x_k^h) - f_k(x_k^a) \\
\alpha (f_k(x_k^l) - f_k(x_k^{\min})) &\neq f_k(x_k^l) - f_k(x_k^a),
\end{aligned}$$

a contradiction. The cases for $z = b$ and $z = c$ follow an analogous argument.

($\Longleftarrow$). By theorem 1, $\succsim_i$ can be represented by a utility function that is linearly decreasing in the city block norm in the space given by some appropriately chosen spatial representation f^i. Since utility functions are rescalable, for each attribute $k \in A$, fix $f_k^{l_k}(x_k^{\max}) - f_k^{l_k}(x_k^{l_k}) = 1$. By linear representability case $z = c$, $\forall x_{-k} \in X_{-k}$, $\forall i \in N$ and for any $x_k^c \in [x_k^i, x_k^{\max}]$, agents l_k and i agree in their preference, hence i also has a utility function that is linearly decreasing in $f_k^{l_k}(x_k^c) - f_k^{l_k}(x_k^i)$. Also note that by linear representability case $z = b$, the utility function of agent h_k is linear in $f_k^{l_k}(x_k^b)$ for any $x_k^b \in [x_k^{l_k}, x_k^{h_k}]$. Construct f_k by letting $f_k(x_k) = f_k^{l_k}(x_k)$ if $x_k \geq_k x_k^{l_k}$ and choosing $f_k(x_k)$ for $x_k \leq_k x_k^{l_k}$ in such manner that the utility of h_k is linear from $f_k(x_k^{\min})$ to $f_k(x_k^{h_k})$. Then, by linear representability case $z = a$, for any $i \in N$ and any $x_k^a \in [x_k^{\min}, x_k^i]$, since agents i and h_k agree in their preference, i also has a utility function that is linearly decreasing in $f_k(x_k^i) - f_k(x_k^a)$ for any fixed $x_{-k} \in X_{-k}$. Therefore, every $i \in N$ has preferences $\succsim_i$ such that given $f_k(x_k)$ and given any $x_{-k} \in X_{-k}$, the utility function $u_i(x)$ that represents $\succsim_i$ is linearly decreasing in $f_k(x_k^i) - f_k(x_k)$ for any $x_k \leq_k x_k^i$ and is linearly decreasing in $f_k(x_k) - f_k(x_k^i)$ for any $x_k \geq_k x_k^i$. But k was arbitrary, so assigning the appropriate weights to each direction in each dimension, and to each dimension, the utility function of each agent in the spatial representation $f(X)$ is linearly decreasing in a generalized weighted city block distance. ■

Proof of proposition 4

Proposition 4 follows as a corollary from theorem 6, for the particular case $\delta = 2$.

Proof of proposition 5

Proof. Given $k \in A$, let l denote l_k and let $v_l(x) \equiv \frac{u_l(x) - u_l(x_k^i, x_{-k})}{u_l(x_k^l, x_{-k}) - u_l(x_k^i, x_{-k})}$ be another utility representation of $\succsim_l$. Note $v_l(x_k^i, x_{-k}) = 0$ and $v_l(x_k^l, x_{-k}) = 1$. Then $(x_k^m, x_{-k}) \sim_l p$ implies $v_l(x_k^m, x_{-k}) = \alpha$. Similarly, let $v_i(x) \equiv \frac{u_i(x) - u_i(x_k^l, x_{-k})}{u_i(x_k^i, x_{-k}) - u_i(x_k^l, x_{-k})}$ be another utility representation of $\succsim_i$. Then $v_i(x_k^l, x_{-k}) = 0$ and $v_i(x_k^i, x_{-k}) = 1$ and $(x_k^m, x_{-k}) \sim_i q$ implies $v_i(x_k^m, x_{-k}) = \alpha$. Without loss of generality, let $f_k(x_k^l) = 0$ and $f_k(x_k^i) = 1$. Given $v_j(x)$ be the utility of j for $j \in \{i, l\}$, let $v_k^j(x_k) = 1 - w_k(x_k, x_k^j)|f_k(x_k) - f_k(x_k^j)|^\delta$ be the utility on attribute k. Notice the change in notation of the utility function, where agent is denoted by a subscript if there is no superscript, but when there is both subscript and superscript, the subscript always denotes the attribute, and the superscript denotes the dimension. Note that $v_l(x_k^l, x_{-k}) - v_l(x_k^i, x_{-k}) = 1$ implies $v_k^l(x_k^l) - v_k^l(x_k^i) = 1$, which implies $w_{k+}^l = 1$. Similarly, $v_i(x_k^i, x_{-k}) - v_i(x_k^l, x_{-k}) = 1$ implies $v_k^i(x_k^i) - v_k^i(x_k^l) = 1$, which implies $w_{k-}^i = 1$. Also note that $v_k^l(x_k^m) = v_k^i(x_k^m) = \alpha$.

In consequence,

$$1 - f_k(x_k^m)^\delta = \alpha = 1 - [1 - f_k(x_k^m)]^\delta$$

Ignoring the middle equality, and looking only at the left hand side and the right hand side,

$$\begin{aligned} f_k(x_k^m)^\delta &= [1 - f_k(x_k^m)]^\delta \\ f_k(x_k^m) &= 1/2. \end{aligned}$$

Then, $1 - f_k(x_k^m)^\delta = \alpha$ implies

$$\begin{aligned} \frac{1}{2^\delta} &= 1 - \alpha \\ \ln \frac{1}{2^\delta} &= \ln(1 - \alpha) \\ -\delta \ln 2 &= \ln(1 - \alpha) \\ \delta &= \frac{-\ln(1 - \alpha)}{\ln 2}. \end{aligned}$$

■

Proof of theorem 6

Proof. ($\Longrightarrow$). Suppose that $u_i(x) = -\sum_{k=1}^{K} w_k^i (f_k(x_k) - f(x_k^i))[f_k(x_k) - f(x_k^i)]^\delta$ for every $i \in N$. Since the utility function depends on the position of points relative to other points, and not on the coordinates of the whole map, we can translate the map. Since utilities do not depend on the size of units, we can also rescale it. Hence, without loss of generality, let $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$. By assumption, $p(x_k^{\max}, x_{-k}) = (\gamma_i)^\delta$ and $p(x_k^{l_k}, x_{-k}) = 1 - (\gamma_i)^\delta$ imply $p \sim_{l_k} (x_k^i, x_{-k})$. Let l be a shorthand notation for l_k. In utility terms,

$$\begin{aligned} \gamma_i^\delta u_l((x_k^{\max}, x_{-k})) + (1 - \gamma_i^\delta) u_l((x_k^l, x_{-k})) &= u_l((x_k^i, x_{-k})). \\ \gamma_i^\delta w_{k+}^l (f_k(x_k^{\max}) - f_k(x_k^l))^\delta &= w_{k+}^l (f_k(x_k^i) - f_k(x_k^l))^\delta \\ \gamma_i &= f_k(x_k^i). \end{aligned}$$

The second equality follows the first because the only difference between the three utility expressions is in dimension k, all others cancel out and the dimension k portion of the second term on the left hand side is zero because x_k^l is the ideal value of l on dimension k. The third equality follows the second because $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$.

In condition i), in utility terms, $p \sim_i (x_k^a, x_{-k}^i)$ if and only if

$$\begin{aligned}
\alpha_i u_i((x_k^{\max}, x_{-k})) + (1-\alpha_i)u_i((x_k^i, x_{-k})) &= u_i((x_k^a, x_{-k})) \\
-\alpha_i w_{k+}^i [f_k(x_k^{\max}) - f_k(x_k^i)]^\delta &= -w_{k+}^i [f_k(x_k^a) - f_k(x_k^i)]^\delta \\
(\alpha_i)^{1/\delta}(1-\gamma_i) &= f_k(x_k^a) - \gamma_i \\
f_k(x_k^a) &= (\alpha_i)^{1/\delta}(1-\gamma_i) + \gamma_i.
\end{aligned}$$

Similarly, $(x_k^a, x_{-k}) \sim_l q$ implies

$$\begin{aligned}
u_l((x_k^a, x_{-k})) &= \alpha_l u_l((x_k^{\max}, x_{-k})) + (1-\alpha_l)u_l((x_k^i, x_{-k})) \\
-w_{k+}^l [f_k(x_k^a) - f(x_k^l)]^\delta &= -\alpha_l w_{k+}^l [f_k(x_k^{\max}) - f_k(x_k^l)]^\delta - (1-\alpha_l) w_{k+}^l [f_k(x_k^i) - f_k(x_k^l)]^\delta \\
(f_k(x_k^a))^\delta &= -\alpha_l - (1-\alpha_l)\gamma_i^\delta \\
\alpha_l + (1-\alpha_l)\gamma_i^\delta &= [(\alpha_i)^{1/\delta}(1-\gamma_i) + \gamma_i]^\delta \\
\alpha_l(1-\gamma_i^\delta) + \gamma_i^\delta &= [(\alpha_i)^{1/\delta}(1-\gamma_i) + \gamma_i]^\delta \\
\alpha_l &= \frac{[(\alpha_i)^{1/\delta}(1-\gamma_i) + \gamma_i]^\delta - \gamma_i^\delta}{1-\gamma_i^\delta}.
\end{aligned}$$

In condition *ii)*, $p \sim_i (x_k^b, x_{-k})$ if and only if

$$\begin{aligned}
(1-\alpha_i){\gamma_i}^\delta &= (\gamma_i - f_k(x_k^b))^\delta \\
f_k(x_k^b) &= \gamma_i - (1-\alpha_i)^{1/\delta}\gamma_i \\
f_k(x_k^b) &= \gamma_i[1-(1-\alpha_i)^{1/\delta}]
\end{aligned}$$

and $q \sim_l (x_k^b, x_{-k})$ implies

$$\begin{aligned}
(1-\alpha_l){\gamma_i}^\delta &= (f_k(x_k^b))^\delta \\
(1-\alpha_l)^{1/\delta}\gamma_i &= \gamma_i[1-(1-\alpha_i)^{1/\delta}] \\
(1-\alpha_l)^{1/\delta} &= 1-(1-\alpha_i)^{1/\delta} \\
1-\alpha_l &= [1-(1-\alpha_i)^{1/\delta}]^\delta \\
\alpha_l &= 1-[1-(1-\alpha_i)^{1/\delta}]^\delta.
\end{aligned}$$

In condition *iii)*, $p \sim_i (x_k^c, x_{-k})$ if and only if

$$\begin{aligned}
\alpha_i u_i((x_k^{\min}, x_{-k})) + (1-\alpha_i)u_i((x_k^i, x_{-k})) &= u_i((x_k^c, x_{-k})) \qquad (1) \\
\alpha_i(\gamma_i - f_k(x_k^{\min}))^\delta &= (\gamma_i - f_k(x_k^c))^\delta.
\end{aligned}$$

By assumption, $q(x_k^{\min}, x_{-k}) = \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^\delta$ and $q(x_k^{h_k}, x_{-k}) = 1 - \left(\frac{\gamma_{h_k}}{\gamma_{h_k}+\gamma_0}\right)^\delta$ imply $q \sim_{h_k} (x_k^l, x_{-k})$. Let h denote h_k. In utility terms,

$$\begin{aligned}
\left(\frac{\gamma_h}{\gamma_h+\gamma_0}\right)^\delta |f_k(x_k^{\min}) - \gamma_h|^\delta &= \gamma_h^\delta \\
\frac{1}{\gamma_h+\gamma_0}(\gamma_h - f_k(x_k^{\min})) &= 1 \\
f_k(x_k^{\min}) &= -\gamma_0.
\end{aligned}$$

Therefore, equation 1 becomes

$$\begin{aligned}
\alpha_i(\gamma_i+\gamma_0)^\delta &= (\gamma_i - f_k(x_k^c))^\delta \\
-(\alpha_i)^{1/\delta}(\gamma_0+\gamma_i)+\gamma_i &= f_k(x_k^c).
\end{aligned}$$

Furthermore, $q \sim_h (x_k^c, x_{-k})$ implies

$$\begin{aligned}
\alpha_h u_h((x_k^{\min}, x_{-k})) + (1-\alpha_h) u_h((x_k^i, x_{-k})) &= u_h((x_k^c, x_{-k})) \\
\alpha_h(\gamma_h + \gamma_0)^\delta + (1-\alpha_h)((\gamma_h - \gamma_i)^\delta &= (\gamma_h + (\alpha_i)^{1/\delta}(\gamma_0 + \gamma_i) - \gamma_i)^\delta \\
\alpha_h[(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta] + (\gamma_h - \gamma_i)^\delta &= (\gamma_h + (\alpha_i)^{1/\delta}(\gamma_0 + \gamma_i) - \gamma_i)^\delta \\
\alpha_h &= \frac{(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)(\alpha_i)^{1/\delta})^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}.
\end{aligned}$$

($\Longleftarrow$). Since the preference $\succsim_i$of every agent i are modular, the utility function $u_i(x)$ that represents $\succsim_i$ is separable and it can be disaggregated into K utility functions, one per dimension, so that $u_i(x) = \sum_{k=1}^{K} u_k^i(x_k)$, as proved by Fishburn [11]. This is the standard separability condition. For an arbitrary $k \in A$ and an arbitrary $x_{-k} \in X_{-k}$, let $l_k, h_k \in N$ be such that $x_{l_k} \leq_k x_i \leq_k x_{h_k}$. If this does not uniquely define l_k, arbitrarily choose of the agents with the lowest ideal value on dimension k and label her l_k. Similarly for h_k. Fix $f_k(x_k^{l_k}) = 0$ and $f_k(x_k^{\max}) = 1$, and for any $x_k \geq_k x_k^{l_k}$, let $f_k(x_k)$ be such that $u_{l_k}((x_k, x_{-k}))$ is linearly decreasing in $|f_k(x_k) - f_k(x_k^{l_k})|^\delta$ for any given $x_{-k} \in X_{-k}$. Then, $f_k(x_k^i) = \gamma_i$, where γ_i is defined by the lottery $p \sim_{l_k} (x^i, x_{-k})$ and $p(x_k^{\max}, x_{-k}) = (\gamma_i)^\delta$ and $p(x_k^{l_k}, x_{-k}) = 1 - (\gamma_i)^\delta$.

To check that $u_i((x_k, x_{-k}))$ is linearly decreasing in $|f_k(x_k) - f_k(x_k^i)|^\delta$ for any $i \in N$ and for any $x_k \geq_k x_k^i$, let $x_k^a \geq x_k^i$ and $p \in \Delta X$ be such that $p(x_k^{\max}, x_{-k}) = \alpha_i, p(x_k^i, x_{-k}) = 1 - \alpha_i$ and $p \sim_i (x_k^a, x_{-k})$, so

$$u_i((x_k^a, x_k^{-k})) = \alpha_i u_i((x_k^{\max}, x_k^{-k})) + (1-\alpha_i) u_i((x_k^i, x_k^{-k})).$$

Since the utilities are separable, every other dimension but k cancels out and

$$u_k^i(x_k^a) = \alpha_i u_k^i(x_k^{\max}) + (1-\alpha_i) u_k^i(x_k^i).$$

Without loss of generality, let $u_k^i(x_k^i) = 0$. Then

$$u_k^i(x_k^a) = \alpha_i u_k^i(x_k^{\max}).$$

Since $f_k(x_k^i) = \gamma_i$ and $f_k(x_k^{\max}) = 1$, in order for $u_k^i(x_k^a)$ to be linearly decreasing in $|f_k(x_k^a) - f_k(x_k^i)|^\delta$ for any $x_k^a \geq_k x_k^i$, we want to show that

$$\begin{aligned} (f_k(x_k^a) - \gamma_i)^\delta &= \alpha_i (1-\gamma_i)^\delta \\ f_k(x_k^a) &= (1-\gamma_i)(\alpha_i)^{1/\delta} + \gamma_i \end{aligned}$$

Since

$$\begin{aligned} q(x_k^{\max}, x_{-k}) &= \frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta - \gamma_i^\delta}{1-\gamma_i^\delta} \text{ and} \\ q(x_k^i, x_{-k}) &= 1 - \frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta - \gamma_i^\delta}{1-\gamma_i^\delta} \end{aligned}$$

together imply $q \sim_{l_k} (x_k^a, x_{-k})$,

$$\begin{aligned} \frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta - \gamma_i^\delta}{1-\gamma_i^\delta} + \left(1 - \frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta - \gamma_i^\delta}{1-\gamma_i^\delta}\right)\gamma_i^\delta &= (f_k(x_k^a))^\delta \\ \left(\frac{\left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta - \gamma_i^\delta}{1-\gamma_i^\delta}\right)(1-\gamma_i^\delta) + \gamma^\delta &= (f_k(x_k^a))^\delta \\ \left(\gamma_i + (1-\gamma_i)\alpha_i^{1/\delta}\right)^\delta &= (f_k(x_k^a))^\delta \\ \gamma_i + (1-\gamma_i)\alpha_i^{1/\delta} &= f_k(x_k^a) \end{aligned}$$

as desired. Hence for every $i \in N$, the utility function on attribute k from x_k^i to $x_k^{\max}$ is linearly decreasing in $|f_k(x_k^a) - f_k(x_k^i)|^\delta$ for any $x_k^a \geq_k x_k^i$.

Similarly, for any $x_k^b \in X_k$ such that $x_k^{l_k} \leq_k x_k^b \leq_k x_k^i$ and $p \in \Delta X$ such that

$p(x_k^i, x_{-k}) = \alpha_i, p(x_k^{l_k}, x_{-k}) = 1 - \alpha_i$ and $p \sim_i (x_k^b, x_{-k})$,

$$\begin{aligned} \alpha_i u_k^i(x_k^i) + (1-\alpha_i) u_k^i(x_k^{l_k}) &= u_k^i(x_k^b) \\ (1-\alpha_i) u_k^i(x_k^{l_k}) &= u_k^i(x_k^b). \end{aligned}$$

Since $f_k(x_k^l) = 0$ and $f_k(x_k^i) = \gamma_i$, we want to show that

$$\begin{aligned} (1-\alpha_i)\gamma_i^\delta &= (\gamma_i - f_k(x_k^b))^\delta \\ (1-\alpha_i)^{1/\delta}\gamma_i &= \gamma_i - f_k(x_k^b) \\ f_k(x_k^b) &= \gamma_i(1-(1-\alpha_i)^{1/\delta}). \end{aligned}$$

Since $q(x_k^l, x_{-k}) = 1 - \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta$, $q(x_k^i, x_{-k}) = \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta$ imply $q \sim_l (x_k^b, x_{-k})$,

$$\begin{aligned} \left(1-(1-\alpha_i)^{1/\delta}\right)^\delta \gamma_i^\delta &= (f_k(x_k^b))^\delta \\ \gamma_i[1-(1-\alpha_i)^{1/\delta}] &= f_k(x_k^b). \end{aligned}$$

Hence the utility function of every agent i is linearly decreasing in $|f_k(x_k^b) - f_k(x_k^i)|^\delta$ for any x_k^b between x_k^l and x_k^i, and, as shown earlier, for any x_k between x_k^i and $x_k^{\max}$. It remains to be shown that the utility function is linearly decreasing in $|f_k(x_k^b) - f_k(x_k^i)|^\delta$ for any $x_k \leq_k x_k^l$. Let h denote h_k. For any $x_k \leq_k x_k^l$, construct $f_k(x_k)$ such that the utility function of h is linearly decreasing in $|f_k(x_k^h) - f_k(x_k)|^\delta$ for any $x_k \leq_k x_k^h$. Then $f_k(x_k^{\min}) = -\gamma_0$. For any $i \in N$ and for any $x_k^c \leq_k x_k^i$, given $p(x_k^{\min}, x_{-k}) = \alpha_i$ and $p(x_k^i, x_{-k}) = 1 - \alpha_i$, if $p \sim_i (x_k^c, x_{-k})$, then

$$\alpha_i u_k^i(x_k^{\min}) = u_k^i(x_k^c).$$

Since $f_k(x_k^i) - f_k(x_k^{\min}) = \gamma_i + \gamma_0$, we want to show

$$\begin{aligned}
\alpha_i(\gamma_i + \gamma_0)^\delta &= (\gamma_i - f_k(x_k^c))^\delta \\
f_k(x_k^c) &= \gamma_i - (\gamma_i + \gamma_0)(a_i)^{1/\delta}
\end{aligned}$$

Given $q(x_k^{\min}, x_{-k}) = \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}$ and $q(x_k^i, x_{-k}) = 1 - \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}$, $(x_k^c, x_{-k}) \sim_h q$ implies

$$\begin{aligned}
(\gamma_h - f_k(x_k^c))^\delta &= \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}(\gamma_h + \gamma_0)^\delta + \\
&\quad \left(1 - \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}\right)(\gamma_h - \gamma_i)^\delta; \\
(\gamma_h - f_k(x_k^c))^\delta &= (\gamma_h - \gamma_i)^\delta + \frac{\left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta - (\gamma_h - \gamma_i)^\delta}{(\gamma_0 + \gamma_h)^\delta - (\gamma_h - \gamma_i)^\delta}[(\gamma_h + \gamma_0)^\delta - (\gamma_h - \gamma_i)^\delta] \\
(\gamma_h - f_k(x_k^c))^\delta &= \left(\gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta}\right)^\delta \\
\gamma_h - f_k(x_k^c) &= \gamma_h - \gamma_i + (\gamma_0 + \gamma_i)\alpha_i^{1/\delta} \\
\gamma_i - (\gamma_0 + \gamma_i)\alpha_i^{1/\delta} &= f_k(x_k^c)
\end{aligned}$$

as desired.

Thus, for every $i \in N$, there exists weights w_{k+} and w_{k-} such that $u_i^k(x_k) = -w_{k+}|f_k(x_k) - f_k(x_k^i)|^\delta$ for every $x_k \geq_k x_k^i$ and $u_i^k(x_k) = w_{k-}|f_k(x_k) - f_k(x_k^i)|^\delta$ for every $x_k \leq_k x_k^i$. Choosing the appropriate relative weights for each dimension k, we obtain $u_i(x) = -\sum_{k=1}^{K} w_k(x_k, x_k^i)|f_k(x_k) - f_k(x_k^i)|^\delta$, where $w_k(x_k, x_k^i) = \{w_{k-}$ if $x_k \leq_k x_k^i$ and w_{k+} if $x_k >_k x_k^i\}$. ■

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