Simultaneous and Sequential Auctions of Oligopoly Licenses

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Abstract

This paper compares two procedures for allocating multiple oligopoly licenses to firms with independent marginal costs: a simultaneous "pay-your-bid" auction and a sequential first-price auction, where the winning bids are announced at the end of each round. When the winners' marginal costs are truthfully revealed after the auction, so that the information their bids convey cannot affect the market competition, the two schemes are allocation and revenue equivalent. In the sequential auction, however, the firms bid in a more informed manner. As a result, the weaker (ex post) of the two oligopolists can win his license at a lower price than the one he would pay in the simultaneous auction. Conversely, the stronger oligopolist pays a higher price. Hence, the sequential auction results in a more equal distribution of the wealth generated by the oligopolistic market. In addition, it eliminates the possibility of winner's regret, that is, of gaining a license at a price that exceeds its eventual value. When the winners' marginal costs have to be inferred from their bids in the auction, it is possible for the firms to enhance their market profits by signaling a different type. In this case, our results extend only in the case of the Cournot oligopoly, in which the firms are better-off overstating their strength. In the Bertrand oligopoly, in which the firms' signaling incentives are negative, a separating equilibrium exists only in the simultaneous auction.

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1 Introduction

Many important auctions have aimed at creating new markets or at expanding markets that already exist. For example, the radio spectrum auctions conducted by the Federal Communications Commission in the United States (and by similar government agencies elsewhere in the world) have allowed the wireless communication companies to expand their services. In the same manner, the auction of other state-owned resources, such as oil fields or timber tracts, have enabled the related companies to expand their operations. In addition, in several regulated markets, companies need to compete periodically for the acquisition or renewal of the license to supply their product. Finally, the development of new technologies often forces those companies that can benefit from them to compete for the acquisition of the right to use them.[1]

Such auctions are typically characterized by the presence of allocative and informational externalities. In particular, the value of the resources or of the licenses that are sold, that is, the market profit that these assets may generate, depends on the entire outcome of the auction. First, it depends on the characteristics of the bidders against which a firm will compete in the market. In addition, it may depend on the identities of the winning and the losing bidders. Finally, when the bidders' characteristics are privately known, their bids convey information about them, in a manner that allows for the possibility of signaling. For example, an oligopoly license is more valuable for its recipient when the firms that gain the other licenses are weaker. Similarly, an oligopolist's overall benefit from acquiring the rights to use a new technology increases when his market competitors, which are prevented from accessing this technology, can derive a greater benefit from it. Finally, in both examples, a firm can profit from exaggerating or from understating its private information during the auction. Hence, because of these externalities, the value of the assets sold is determined endogenously, by the types of the bidders, the final allocation and the information that the auction reveals.[2]

In this paper, we study a particular environment in which such externalities are present: an auction of two licenses to supply in a Cournot oligopoly or in a Bertrand oligopoly for differentiated products.[3] For each of the competing firms, the value of the auctioned

[1]For a survey of the FCC auctions, see Cramton [5, 6]; for a survey of similar auctions in Europe, see Jehiel and Moldovanu [21] and Klemperer [32]. For information regarding the auctions of oil sources or of timber tracts, see, respectively, Cramton [7] and Hendricks and Porter [15] as well as the references therein. For the relation between an auction and the market created by it, see Dana and Spier [8]. Regarding patent licensing schemes, see Kamien [27], Kamien et al. [28] and Kamien and Tauman [29]. Finally, for a general survey of auctions of public assets and the issues that typically arise in them, see Janssen [18].

[2]This environment differs from that of an auction with symmetric interdependent valuations, such as the one studied in Milgrom and Weber [35]. There, the value of the assets sold depends on the bidders' private information only. It is independent of the outcome of the auction; in particular, it does not depend on who wins the auction or on the information that the auction may reveal.

[3]For simplicity, we have assumed that the sale of the licenses results in the creation of a new market. Our results, however, easily extend to an auction of two licenses to enter an already existing oligopoly or monopoly. In such an auction, the structure of the firms' payoffs and incentives is identical to that in our model. Similarly, one can modify our model so as to describe the auction of two licenses to use a process innovation. Again, the structure of the firms' incentives will not change.

licenses depends, primarily, on its own production costs. In addition, it depends on the production costs of the winner of the other license, that is, of the firm with which it will compete in the market. Finally, when the firms' production costs cannot be revealed otherwise, there is the possibility of signaling through bidding. In a Cournot oligopoly, a firm can increase its market profit by signaling, during the auction, a stronger type, so as to beguile its opponent into supplying a smaller quantity. Similarly, in a Bertrand oligopoly, a firm can increase its profit by signaling a weaker type, so as to lure its opponent into setting a higher price.

We compare two procedures for allocating the oligopoly licenses, a simultaneous "pay-your-bid" auction[4] and a sequential first-price auction. We assume that in each auction scheme, the seller reveals the same information, namely, the two winning bids. The two schemes differ, however, in the timing of revelation of that information. In the simultaneous format, all information is revealed at the end of the entire procedure, prior only to market competition. On the other hand, in the sequential format, some information, the bid that won the first license, is revealed during the bidding process, in between the two auction rounds. As a result, in the sequential auction, the firms can bid in a better-informed manner, as some of the strategic uncertainty present in this environment is eliminated.

When the winners' production costs are truthfully revealed after the auction[5], so that the information conveyed by their bids does not affect the ensuing market competition, then for each auction scheme there exists a symmetric equilibrium in strictly monotone bidding strategies. The two licenses are therefore allocated to the two strongest firms. The difference in the information structure, however, affects the degree by which the firms shade their bids and, therefore, the prices that they eventually pay.

In the first round of the sequential auction, the firms know that they will win the license only if they have the strongest type. Therefore, since they do not take into account the possibility of having to compete against a stronger market opponent, they shade their bids less than they would do in the simultaneous auction. Consequently, the stronger of the two market competitors, as revealed by the outcome of the auction, pays a higher price in the sequential auction than in the simultaneous one. Similarly, the firms participating in the second round know that, by winning the second license, they will necessarily have to face a stronger market competitor. Therefore, on average, they shade their bids more than they would do in the simultaneous auction. As a result, the weaker of the two market competitors pays a higher price in the simultaneous auction.[6]

[4] In Krishna [33] and elsewhere, this auction is referred to as a discriminatory auction.

[5] The revelation of the two oligopolists' production costs can be the consequence of the actions that they need to take in the time period between the end of the auction and the beginning of the market. Truthful revelation can also be assumed when the effects of false signaling are negligible, for example, when the oligopolists can quickly adjust their market strategies. This assumption is present in much of the literature on auctions with externalities, for example, in Jehiel and Moldovanu [20].

[6] In addition, in the simultaneous auction, the weaker of the two oligopolists may gain his license at a price that exceeds its revealed value, thus, regretting his participation to the market. Such a negative outcome, however, cannot occur in the sequential auction.

Despite the differences in the bidders' behavior, the two auction schemes turn out to be revenue equivalent.[7] The excess of aggression that the bidders show in the first round of the sequential auction, which stems from the good news that a victory in this round will convey, is balanced, on average, by the excess of restraint that the bidders show in the second round, in reaction to the bad news they have received. Therefore, without incurring any cost to the seller, the sequential auction results in a more even distribution of the wealth generated by the introduction of the new market.[8]

When the oligopolists' production costs are not automatically revealed after the auction, but have to be inferred from their bids, the possibility of signaling is introduced. In this case, the firms adjust their valuations by incorporating the informational rents that they can extract. In each auction scheme, there are two incentive trade-offs, the signaling and the non-signaling one. Since the firms' market profit functions are separable in their real and signalled production costs, the two trade-offs can be separated and treated independently. Therefore, the firms' non-signaling incentives can always be balanced, as in the case in which their costs are truthfully revealed. On the other hand, the possibility of balancing the signaling incentives depends on the auction scheme as well as on the type of the market.

In the simultaneous auction, the firms can adjust their strategies so that any gains from false signaling can be offset by an increase in the expected payment, in the case of the Cournot market, or by a decrease in the probability of winning a license, in the case of the Bertrand market. In the sequential auction, a similar trade-off is possible only in the Cournot oligopoly. In the Bertrand oligopoly, a firm can always profit from mimicking a weaker type in the first round, as this would increase its expected market profit and decrease its expected payment, without changing its overall probability of winning one of the two licenses. Therefore, in this case, no symmetric monotone equilibrium exists.

Hence, in the Cournot oligopoly, the non-signaling results remain valid under signaling. The two auction schemes are revenue equivalent, even though the sequential auction favors the weaker (ex-post) of the two oligopolists. On the other hand, in the Bertrand oligopoly, a comparison between the two auction schemes is not possible, since a separating equilibrium exists only in the simultaneous auction.[9]

[7] Since we assume that the firms' production costs are independent, the intuition of the Linkage Principle for auctions with interdependent valuations (cf. Milgrom and Weber [35]) does not apply to our setting. For this principle to impose a revenue ranking in favor of the auction scheme that reveals more information during the bidding procedure, in this case, in favor of the sequential auction, the firms' production costs must be affiliated.

[8] The seller's concern for a more even distribution of the oligopolists' overall profit may stem from a desire to maintain balance, over time, among the competing firms. In this case, therefore, the choice of a sequential auction can be thought off as an indirect subsidy to the weaker market participant. For a discussion of this issue, see Maasland et al. [34].

[9] In light of the results regarding information sharing in oligopoly (cf. Gal-Or [11, 12]), our signaling results are hardly surprising. In particular, when signaling is possible, a separating equilibrium exists only in the oligopoly in which the firms are willing to share information about their production costs.

Overall, our results regarding the comparison of the two auction schemes are based on the ability of the sequential auction to generate implicitly, by the characterization of the equilibrium, information about the two winners' relative strengths. This information enables the firms to modify their interim valuations for each of the auctioned licenses. The first license becomes more valuable, relative to a license won in a simultaneous auction, since its acquisition implies a stronger presence in the market. On the other hand, the second license becomes less valuable, since its recipient will have a weaker market presence. As a result, the licenses (and the information that accompanies their acquisition) are sold at different prices than the ones paid in the simultaneous auction.

The early study of auctions with externalities[10] assumed that the externalities depend only on the number of allocated objects (and not on the bidders' types and identities). In their study of the persistence of a monopoly, Gilbert and Newbery [13] show that a monopolist who faces a potential entrant may bid for a technological innovation for which he has no use, in a preemptive manner. Within the context of patent licensing and vertical contracting, Katz and Shapiro [30], Kamien and Tauman [29] and Kamien et al. [28] compare some typical licensing mechanisms, such as auctions, fixed fees and royalties, and show the superiority of auctions.

Optimal selling mechanisms in the presence of type-dependent externalities were first studied by Jehiel et al. [25, 26]. If the agents' private information is multi-dimensional, then it is optimal for the seller to employ identity-dependent "threats", which exploit the negative allocative externalities in order to extract surplus from bidders that do not acquire the license. For very strong negative externalities, the bidders may even pay the seller not to allocate the license at all. Clearly, this mechanism is not efficient. In fact, Jehiel and Moldovanu [22, 23] show that, with multi-dimensional signals, efficiency cannot be implemented. If the agents' private information is single-dimensional, however, efficiency is feasible. Figueroa and Skreta [10] show that sometimes the optimal mechanism is efficient; at other times, though, it allocates the auctioned objects in a random manner.

Since the application of the optimal mechanism, in particular, the differential treatment of the bidders, may not be possible, Jehiel and Moldovanu [19, 20] examine relatively simpler selling schemes, such as auctions with fixed reserve prices or entry fees. They find that some bidders may prefer to abstain from the auction, if their participation can have an adverse effect on bidding or on the winner's identity. Conversely, to encourage participation, the seller may set a reserve price below his own reservation value. In a multi-unit setting, in particular, in the sale of the rights to use a cost-reducing innovation in an oligopoly, Schmitz [37] and Bagchi [2] examine simultaneous auctions of a predetermined number of licenses. As the bidders' information is single-dimensional, the licenses are allocated efficiently. In addition, the seller can be better off auctioning multiple licenses rather than the exclusive rights to use the technology. Finally, in an auction of multiple licenses to enter an already existing oligopoly, Hoppe et al. [16] show that the resulting market can be less competitive if more licenses are made available.

[10] For an extensive survey of the literature on this subject, see Jehiel and Moldovanu [24].

Signaling in auctions with externalities was introduced in Goeree [14], who examined the auction of a single license to compete against a monopolist with known marginal cost. Das Varma [9], in a problem of bidding for the acquisition of a cost-reducing patent, identified conditions for the existence of equilibrium in the presence of negative informational externalities. Katzman and Rhodes-Kropf [31] extended the study of signaling to more general schemes of information revelation, showing the revenue equivalence of the auction schemes that result to the same allocation and reveal the same information. Finally, Molnár and Virág [36] determine the revenue maximizing allocation and information mechanism in environments with post-auction interaction.

The present work contributes to the existing literature on auctions with externalities by extending the analysis of the multi-unit case to sequential auctions. When there is no signaling, we show that both the simultaneous and the sequential scheme lead to an efficient allocation of the licenses[11] while they raise the same revenue for the seller. In addition, we demonstrate the effects of the better-informed bidding that is allowed by the sequential format, showing that it favors the weaker, ex-post, of the two oligopolists. Hence, in the absence of informational externalities, we conclude that the sequential auction can be recommended as a policy device to an auctioneer who wishes to achieve a more even distribution of the wealth generated by the creation of the oligopoly. Finally, we explore the implications of signaling, showing that the incentive to understate one's strength, which is present in the case of the Bertrand oligopoly, eliminates the possibility of efficient allocation.

In the next two sections, we present the model describing our problem and we analyze the firms' behavior in the oligopoly created by the auction of the two licenses. In section 4, we examine the two auction procedures when there is no signaling, deriving symmetric equilibria in strictly monotone bidding strategies and comparing them. In section 5, we analyze the case of positive signaling, which is present in the Cournot oligopoly, showing that the non-signaling results fully extend. In section 6, we study the case of negative signaling, present in the Bertrand oligopoly, showing that a symmetric equilibrium in strictly monotone strategies exists only for the simultaneous auction. Finally, we conclude in section 7.

[11] Since the bidders' private information is single-dimensional, the efficiency result in our setting does not contradict Jehiel and Moldovanu [22, 23].

2 General Model

We study the auction of 2 licenses for participating in a newly formed oligopoly, which will take the form of either Cournot competition or Bertrand competition for differentiated products. The market profits of the two oligopolists depend, respectively, on the quantities they supply to the market or on the prices they set for their product. These decisions depend, in turn, on the oligopolists' production costs.

There are $N > 2$ firms competing for the acquisition of the oligopoly licenses. The firms have linear production technologies without fixed costs. Therefore, for each firm i, its technology is characterized by the privately known marginal cost c_i, which is drawn independently, at the beginning of the game, from a distribution $F : [\underline{c}, \bar{c}] \to [0, 1]$, where $0 < \underline{c} < \bar{c} < \frac{1+\underline{c}}{2}$.[12] We assume that F is twice differentiable, with a density function $f : [\underline{c}, \bar{c}] \to \mathbb{R}^+$ that has full support. Finally, we assume that the inverse hazard rate $[1 - F(c)]/f(c)$ is decreasing.[13]

For any firm i, we denote by c^1_{-i} and c^2_{-i} the random variables describing respectively the lowest and the second-lowest marginal costs of firm i's competitors (and by c^1 and c^2 the values that these random variables may take). In addition, we denote by $G(c^1) = 1 - [1 - F(c^1)]^{N-1}$ the cumulative distribution function of c^1_{-i} and by

$$g(c^1) = (N - 1)\,[1 - F(c^1)]^{N-2}\, f(c^1)$$

the corresponding density function. Finally, we denote by $G(c^1, c^2)$, for $c^1 \le c^2$, the joint cumulative distribution function of c^1_{-i} and c^2_{-i} and by

$$g(c^1, c^2) = (N - 1)(N - 2)\,[1 - F(c^2)]^{N-3}\, f(c^2)\, f(c^1),$$

for $c^1 \le c^2$, the joint density function.

We consider two auction formats:

a. A simultaneous pay-your-bid auction, with the two winning bids announced at the end of the auction.

b. A sequence of two first-price auctions, with the winning bid announced at the end of each auction.

In both formats, the winners' bids are publicly known by the end of the auction process, so that the information they reveal affects the ensuing market competition. Furthermore, in the sequential auction, the winning bid in the first round becomes known prior

[12] In particular, this last assumption implies that $\bar{c} < 1$.

[13] This assumption is satisfied by many well-known distributions, such as the uniform, exponential, normal, power (for $\alpha \ge 1$), Weibull (for $\alpha \ge 1$) and gamma (for $\alpha \ge 1$) distributions. It is consistent with the assumption of logconcave distribution of the firms' strength, made elsewhere in the literature, in particular, in Das Varma [9] and Goeree [14]. If the firms' strength $\theta \in [\underline{c}, \bar{c}]$, defined by $\theta(c) = (\underline{c}+\bar{c})-c$, is distributed according to a logconcave density $\tilde{f}$, then the rate $\tilde{F}(\theta)/\tilde{f}(\theta)$ must be increasing. This, in turn, implies that the hazard rate $[1-F(c)]/f(c)$ must be decreasing. For the definition and properties of logconcave probability density functions, consult An [1] and Caplin and Nalebuff [4]; for more on the assumption of increasing inverse hazard rate, consult Hoppe et al. [17].

to the beginning of the second round, so that the information it conveys also affects the bidding for the second license. Since there are no reserve prices in any of the auctions, we can assume that the two licenses are always sold, even at a zero price.

We will restrict attention to equilibria in symmetric strategies, strictly monotone in the firm's own marginal cost. Therefore, in the simultaneous auction, each firm i bids $b_i = \beta(c_i)$, according to its marginal cost c_i and the strategy

$$\beta : [\underline{c}, \bar{c}] \longrightarrow \mathbb{R}^+.$$

Similarly, in the sequential auction, each firm i bids $b_i^1 = \beta^1(c_i)$ in the first round, according to its marginal cost c_i and the strategy

$$\beta^1 : [\underline{c}, \bar{c}] \longrightarrow \mathbb{R}^+.$$

If it fails to win the first round, then firm i bids $b_i^2 = \beta^2(c_i \,|\, b_i^1, b^1)$ in the second round, according to its marginal cost c_i, the first-round history

$$h_i^1 = (b_i^1, b^1) \;\in\; H^1_{\text{sqc}} = \mathbb{R}^+ \times \mathbb{R}^+,$$

consisting of the privately known bid b_i^1 and the publicly known price b^1, and the strategy

$$\beta^2 : [\underline{c}, \bar{c}] \times H^1_{\text{sqc}} \longrightarrow \mathbb{R}^+.$$

Following the completion of the auction, each of the winning firms enters the oligopoly. The information that firm i has at the end of the simultaneous auction,

$$h_i = (b_i, b^1, b^2) \;\in\; H_{\text{sim}} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,$$

consists of its privately known bid b_i and the publicly known prices b^1, b^2.[14] Similarly, the information that firm i has at the end of the sequential auction is

$$h_i^2 = (b_i^1, b_i^2, b^1, b^2) \;\in\; H^2_{\text{sqc}} = \mathbb{R}^+ \times \big(\mathbb{R}^+ \cup \{\emptyset\}\big) \times \mathbb{R}^+ \times \mathbb{R}^+,$$

allowing for the absence of a second-round bid, in the case of a first-round win.

Given this information and its marginal cost c_i, firm i will supply $q_i = q(c_i \,|\, b_i, b^1, b^2)$ or $q_i = q(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$ in the Cournot oligopoly, according to the strategy

$$q : [\underline{c}, \bar{c}] \times H \longrightarrow \mathbb{R}^+,$$

for $H \in \{H_{\text{sim}},\, H^2_{\text{sqc}}\}$, following respectively a simultaneous or a sequential auction.

Similarly, in the Bertrand oligopoly, firm i will set a price $p_i = p(c_i \,|\, b_i, b^1, b^2)$ or $p_i = p(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$, according to the strategy

$$p : [\underline{c}, \bar{c}] \times H \longrightarrow \mathbb{R}^+,$$

[14] Without loss of generality, the two prices are in descending order.

for $H \in \{H_{\text{sim}},\, H^2_{\text{sqc}}\}$, following respectively a simultaneous or a sequential auction.

Overall, we will impose a stronger symmetry requirement, one that rules out the possibility of using past histories as a labeling device for asymmetric continuation strategies. This assumption will rule out, in particular, asymmetric supply or price-setting strategies for the two oligopolists.[15]

In the sequel, we will use the strict monotonicity of the bidding strategies, with respect to the firm's own marginal cost, to simplify the notation in the following manner:

Notation:
In the sequential auction, we will denote[16] *firm i's second-period bid* b_i^2*, following a first-period bid* b_i^1 *and a price* $b^1 = \beta^1(c^1)$*, by*

$$\beta^2(c_i \,|\, b_i^1, \beta^1(c^1)) \;\equiv\; \beta^2(c_i \,|\, c^1).$$

In the Cournot oligopoly, following either a simultaneous or a sequential auction, we will denote firm i's supplied quantity, $q_i = q(c_i \,|\, b_i, b^1, b^2)$ *or* $q_i = q(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$*, by*

$$q_i \;\equiv\; q(c_i \,|\, \tilde{c}_i, \tilde{c}_j),$$

where $\tilde{c}_i$ *and* $\tilde{c}_j$ *are the marginal costs corresponding to the bids, under the equilibrium bidding strategies, with which firms i and j won their oligopoly licenses.*

Similarly, in the Bertrand oligopoly, following either of the two auction formats, we will denote firm i's requested price, $p_i = p(c_i \,|\, b_i, b^1, b^2)$ *or* $p_i = p(c_i \,|\, b_i^1, b_i^2, b^1, b^2)$*, by*

$$p_i \;\equiv\; p(c_i \,|\, \tilde{c}_i, \tilde{c}_j).$$

The notational simplification of the oligopoly supply or price setting strategies is also based on the independence of these strategies of the firm's privately known bids. Indeed, as it will turn out, each oligopolist's behavior depends only on its own marginal cost, c_i, its opponent's inferred marginal cost, $\tilde{c}_j$, and its own marginal cost as perceived by its opponent, $\tilde{c}_i$. Since the costs $\tilde{c}_i$ and $\tilde{c}_j$ are inferred by the publicly known prices, the privately known bids provide no information.

The game payoff of firm i, in case it wins a license, equals its profit from the oligopoly minus the price that it paid for the license. Otherwise, if it does not win any license, it equals zero.

The solution concept is that of perfect Bayesian equilibrium. The players must therefore behave optimally at each decision point, given their knowledge of the other players' strategies and their beliefs. On the equilibrium path, these beliefs are formed by applying Bayes' rule while, off the equilibrium path, they are arbitrary.

[15] For example, in the sequential setting, this assumption does not allow the possibility of prescribing different oligopoly strategies to the winners of the first and the second sequential auctions.

[16] This simplification is customary in sequential auctions; for example, see Krishna [33], chapter 15. It is based on the strict monotonicity of the equilibrium bidding strategy β^1 as well as on the independence of the equilibrium bidding strategy β^2 of the first-round bid b_i^1.

3 Market Competition

When the two winners' marginal costs, c_i and c_j, are truthfully revealed at the end of the auction, the firms cannot use their bids to manipulate their market opportunities, that is, no signaling is possible. In this case, each oligopolist supplies a quantity $q_i = q^{NS}(c_i, c_j)$ or sets a price $p_i = p^{NS}(c_i, c_j)$, which is independent of the bids submitted to or the prices reported in the auction.

When the oligopolists must infer their opponent's marginal cost by the reported prices, the opportunity of signaling arises. In this case, the two winning bids b_i^t and $b_j^{t'}$ perfectly reveal, through the inversion of the corresponding strategies $\beta^t(.)$ and $\beta^{t'}(.)$, the marginal costs $\tilde{c}_i$ and $\tilde{c}_j$ that the winners mimicked in the auction.[17] Therefore, each firm supplies $q_i = q(c_i \,|\, \tilde{c}_i, \tilde{c}_j)$ or sets a price $p_i = p(c_i \,|\, \tilde{c}_i, \tilde{c}_j)$. Since we consider only unilateral deviations, in examining the incentives of player i we will assume that $\tilde{c}_j = c_j$, so that $q_i = q(c_i \,|\, \tilde{c}_i, c_j)$ and $p_i = p(c_i \,|\, \tilde{c}_i, c_j)$.

Clearly, in the equilibrium path, the two firms reveal their marginal costs truthfully. As a result, for all $c_i, c_j \in [\underline{c}, \bar{c}]$, we have $q(c_i \,|\, c_i, c_j) = q^{NS}(c_i, c_j)$ and $p(c_i \,|\, c_i, c_j) = p^{NS}(c_i, c_j)$. In analyzing the firms' market behavior, therefore, we will consider only the case of signaling, treating the absence of signaling as one of its particular outcomes.

3.1 Cournot Oligopoly

We consider a Cournot oligopoly, in which the inverse demand function is given by $p = 1 - q$, where p is the market price and $q = q_i + q_j$ is the aggregate supply of oligopolists i and j.

If firm i reveals its marginal cost c_i truthfully, then, by supplying $q_i \in [0, 1 - q_j]$ in response to $q_j \in [0, 1]$, it will make a market profit

$$\pi(q_i, q_j) \;=\; q_i\ (1 - q_i - q_j - c_i).$$

Therefore, in equilibrium, firm i will supply[18]

$$q(c_i \,|\, c_i, c_j) \;=\; \tfrac{1}{3}\ (1 + c_j - 2c_i),$$

for a profit of

$$\pi(c_i \,|\, c_i, c_j) \;=\; (\tfrac{1}{3})^2\ (1 + c_j - 2c_i)^2.$$

[17] In the sequential auction, it is possible for a bidder to deviate into mimicking two different types, if he does not win in the first round. In this case, however, only the type mimicked in the second auction will be revealed and, therefore, be relevant in the analysis of the post-auction competition.

[18] The assumption of $\bar{c} < \frac{1+\underline{c}}{2}$ ensures that the market does not become a monopoly. In equilibrium, it is optimal for a firm to supply a positive quantity to the market, regardless of its own marginal cost and its beliefs about the marginal cost of the other firm.

Off the equilibrium path, if firm i mimics a type $\tilde{c}_i \neq c_i$ in the auction that it wins, firm j will supply $q(c_j \,|\, c_j, \tilde{c}_i) \;=\; \frac{1}{3}\,(1 + \tilde{c}_i - 2c_j)$. Therefore, firm i will maximize its profit by supplying

$$q(c_i \,|\, \tilde{c}_i, c_j) \;=\; \tfrac{1}{3}\;(1 + c_j - \tfrac{3}{2}c_i - \tfrac{1}{2}\tilde{c}_i).$$

In this case, its market profit will be

$$\pi(c_i \,|\, \tilde{c}_i, c_j) \;=\; (\tfrac{1}{3})^2\,(1 + c_j - \tfrac{3}{2}c_i - \tfrac{1}{2}\tilde{c}_i)^2.$$

Clearly, we have

$$\pi_2 \;=\; \frac{\partial \pi}{\partial \tilde{c}_i} \;<\; 0,$$

so, prior to market competition, during the auction process, each firm has an incentive to overstate its power by mimicking a lower marginal cost.

3.2 Bertrand Oligopoly

We consider a Bertrand oligopoly, in which each firm i faces a linear demand function $q_i = 1 - p_i + \gamma\, p_j$, where p_i and p_j are the prices set respectively by firm i and its rival, firm j, while $\gamma \in [0, 1)$ is a parameter reflecting the degree of product differentiation.[19] Since $\gamma < 1$, the demand faced by each firm is more responsive to changes in the price charged by this firm than to changes in the price charged by its rival.

If firm i reveals its marginal cost c_i truthfully, then, by setting a price $p_i \in [0, 1 + \gamma\, p_j]$ in response to a price $p_j \in [0, 1]$, it will make a market profit

$$\pi(p_i, p_j) \;=\; (1 - p_i + \gamma\, p_j)\;(p_i - c_i).$$

Therefore, in equilibrium, firm i will set a price

$$p(c_i \,|\, c_i, c_j) \;=\; \frac{2 + \gamma + \gamma\, c_j + 2c_i}{4 - \gamma^2}\,,$$

for a profit of

$$\pi(c_i \,|\, c_i, c_j) \;=\; \frac{[2 + \gamma + \gamma\, c_j - (2 - \gamma^2)\, c_i]^2}{(4 - \gamma^2)^2}\,.$$

Off equilibrium, if firm i mimics a type $\tilde{c}_i \neq c_i$, firm j will set a price $p_j = p(c_j \,|\, c_j, \tilde{c}_i)$. Therefore, firm i will be best-off by setting a price

$$p(c_i \,|\, \tilde{c}_i, c_j) \;=\; \frac{2(2 + \gamma) + 2\gamma\, c_j + (4 - \gamma^2)\, c_i + \gamma^2\, \tilde{c}_i}{2(4 - \gamma^2)}\,,$$

[19] Our results will not change, if we consider a more general linear demand function $q_i = \alpha - \beta\, q_i + \gamma\, p_j$, for $\beta \geq \gamma$, as the induced equilibrium market profit function will demonstrate the same properties.

for a market profit of

$$\pi(\tilde{c}_i \,|\, c_i, c_j) \;=\; \frac{[\,2(2+\gamma) + 2\gamma\, c_j - (4-\gamma^2)\, c_i + \gamma^2\, \tilde{c}_i\,]^2}{4\,(4-\gamma^2)^2}\,.$$

In this market, we have

$$\pi_2 \;=\; \frac{\partial \pi}{\partial \tilde{c}_i} \;>\; 0,$$

so, during the auction process, each firm has an incentive to understate its power by mimicking a higher marginal cost.

3.3 General Remarks

For both oligopolies, the market profit function of each firm i is decreasing in its own marginal cost c_i and increasing in its rival's marginal cost c_j. That is,

$$\pi_1 \;=\; \frac{\partial \pi}{\partial c_i} < 0$$

and

$$\pi_3 \;=\; \frac{\partial \pi}{\partial c_j} > 0.$$

Furthermore, for all $c \in [\underline{c}, \bar{c}]$, we have

$$\frac{d}{dc}[\pi(c \,|\, c, c)] \;=\; \pi_1(c \,|\, c, c) + \pi_2(c \,|\, c, c) + \pi_3(c \,|\, c, c) \;<\; 0,$$

so that, with truthful revelation of the firms' marginal costs, a firm's market profit will be affected more by a change in its own marginal cost than by the same change in its rival's marginal cost.

The two oligopolies differ in the sign of the derivative

$$\pi_2 = \frac{\partial \pi}{\partial \tilde{c}_i},$$

that is, in the signaling incentives of the firms. In the Cournot oligopoly, there is positive signaling, that is, each firm has an incentive to overstate its power. On the other hand, in the Bertrand oligopoly, there is negative signaling, that is, each firm is better off understating its power. Other than that, in particular, when there is no signaling, the profit functions in the two oligopolies induce the same, qualitatively, incentives.[20]

[20]The opposite signaling incentives correspond to the distinction between strategic substitutes and strategic complements, introduced by Bulow et al. [3]. In particular, in the Cournot oligopoly that we have described, quantities are strategic substitutes, since an increase in q_i causes a decrease in the q_j (as well as a decrease in the profit of firm j). On the other hand, in the Bertrand oligopoly, prices are strategic complements, since an increase in p_i causes an increase in p_j (and in the profit of firm j).

4 No Signaling

When signaling is not possible, the value of each oligopoly license,

$$\pi^{NS}(c_i, c_j) \;=\; \pi(c_i | c_i, c_j),$$

is fully determined by the actual marginal costs of the two firms that compete in the market. Mimicking a different type during the auction process cannot affect a firm's potential market profits. It only affects the firm's probability of winning the auction and its expected payment in it.

4.1 Simultaneous Auction

Suppose that all firms follow a strictly decreasing bidding strategy $b = \beta(c)$ and consider firm i with marginal cost c_i. Then, by mimicking a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ during the auction, firm i will win a license if and only if $\tilde{c}_i \leq c^2_{-i}$. In this case, the actual value of this license will be equal to firm i's market profit, $\pi^{NS}(c_i, c^1_{-i})$, which depends on the marginal cost c^1_{-i} of the winner of the other license. Therefore, for a bid $\tilde{b}_i = \beta(\tilde{c}_i)$, the expected total payoff of firm i is

$$\Pi(\tilde{c}_i \,|\, c_i) \;=\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i] \times [\, \mathbb{E}_{c^1_{-i}}[\pi^{NS}(c_i, c^1_{-i}) \,|\, c^2_{-i} \geq \tilde{c}_i\,] \,-\, \beta(\tilde{c}_i)\,]$$

or, by expanding the term for the firm's expected market profit,

$$\begin{aligned}
\Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\,(N-1)[1 - F(\tilde{c}_i)]^{N-2} f(c^1)\, dc^1 \\
&+\; \int_{\tilde{c}_i}^{\bar{c}} \pi^{NS}(c_i, c^1)\,(N-1)[1 - F(c^1)]^{N-2} f(c^1)\, dc^1 \\
&-\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\; \beta(\tilde{c}_i).
\end{aligned}$$

The first-order condition with respect to $\tilde{c}_i$ results in the equation

$$\frac{d}{d\tilde{c}_i}\{\mathbb{P}[c^2_{-i} \geq c_i]\, \beta(c_i)\} \;=\; -\int_{\underline{c}}^{c_i} \pi^{NS}(c_i, c^1)\,(N-1)(N-2)\,[1 - F(c_i)]^{N-3}\, f(c_i) f(c^1)\, dc^1,$$

which requires, in a manner that is standard for "pay-your-bid" auction schemes, that any increase in the firm's expected market profit from a deviation to $\tilde{c}_i \neq c_i$ must be offset by an increase in the firm's expected payment in the auction.

The differential equation derived from the first-order condition, along with the boundary condition expressing the equilibrium behavior of the weakest type $c_i = \bar{c}$,

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

which guarantees the uniqueness of the solution to that differential equation, provides the equilibrium bidding strategy for this setting.

Proposition 1:
In the simultaneous pay-your-bid auction of two oligopoly licenses, in which the winners' marginal costs are revealed truthfully after the auction, the following strategy constitutes a symmetric separating equilibrium:

$$\beta(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi^{NS}(c^2, c^1)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, f(c^2) f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2.$$

Proof: Appendix.

The strategy $\beta(c_i)$ can also be expressed as

$$\beta(c_i) \;=\; \int_{c_i}^{\bar{c}} v^{NS}(c^2)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, F(c^2) f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^2,$$

where

$$v^{NS}(c) \;=\; \int_{\underline{c}}^{c} \pi^{NS}(c, c^1)\, \frac{f(c^1)}{F(c)}\, dc^1,$$

for $c \in [\underline{c}, \bar{c}]$, is the expected market profit of a firm with marginal cost c, assuming that its market opponent is stronger. Therefore, in equilibrium, each firm submits a bid equal to the expected market profit of the strongest non-winning firm.

The value of the licences that the two winners of the auction gain is determined endogenously, as a function of the marginal costs of the winning bids. Since these costs are unknown prior to the end of the auction process, it is possible for a firm, when its market opponent turns out to be stronger than expected, to acquire a licence at a price above its ex-post value.[21]

[21] We emphasize the difference between this phenomenon and the winner's curse for an auction with interdependent valuations (as well as for our environment). The winner's curse refers to the bad news that a victory in such an auction conveys, namely, that the winner's estimate of the value of the auctioned object has been the most optimistic one. In the equilibrium path, the winner's curse is eliminated by means of an adjustment of the bidders' estimates. Still, it is possible that the losing bidders' private information will be very negative, so as to defy the winner's reasonable expectation and to result in a value that is below the price the winner must pay. It is this phenomenon to which we refer as the winner's regret.

Corollary 1:
In the simultaneous auction, the firm with the lowest marginal cost gains a license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain a license at a price above its ex-post value.

Proof: Appendix.

Hence, in equilibrium, the stronger of the two oligopolists will always make a positive profit. On the other hand, the weaker oligopolist may regret his participation to the market, because of the price of the license.

4.2 Sequential Auction

When the two licenses are allocated by means of a sequence of first-price auctions, then, assuming that the firms follow strictly monotone bidding strategies, the winning bid in the first auction reveals the marginal cost c^1 of the strongest oligopolist. This information affects the bidding for the second license in two distinct manners. First, it allows the remaining firms to learn, prior to the second auction, the actual value of the license for which they compete. In addition, the revealed marginal cost c^1 forms a lower bound for the marginal costs of the remaining firms. Therefore, after the end of the first auction, the firms update their beliefs, so that

$$c_i \sim \tilde{F}(c) = \frac{F(c) - F(c^1)}{1 - F(c^1)},$$

for $c \in [c^1, \bar{c}]$.

Since the privately known first-period bids do not affect the firms' behavior in the second period, the second auction takes the form of a standard first-price auction with independent private values.

Lemma 1:
Suppose that $N-1$ firms, whose marginal costs are i.i.d. according to the distribution function $F(\cdot)$ on $[\underline{c}, \bar{c}]$, compete in a first-price auction for a license to participate in an oligopoly against a firm with known marginal cost $c^1 \in [\underline{c}, \bar{c}]$. In addition, suppose that the firms believe that the unknown marginal costs are bounded below by the value c^1. Then, assuming that the winner's marginal cost is revealed truthfully at the end of the auction, the following strategy constitutes a symmetric equilibrium:

For a marginal cost $c_i \geq c^1$, firm i bids

$$\beta^2(c_i \,|c^1) \;=\; \int_{c_i}^{\bar{c}} \pi^{NS}(c, c^1)\, \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc,$$

while for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1 \,|c^1)$.

A marginal cost $c_i < c^1$ corresponds to an event off the equilibrium path, namely, to the case in which firm i should have won the first license but did not bid according to the strategy β^1 that was prescribed in the first auction.[22] Such a firm enters the second auction knowing that it has the highest valuation and it would be best-off bidding as if it had marginal cost c^1.

For the analysis of the firms' behavior in the first auction, we will need the strategy $\beta^2(c_i|c_i)$ to be decreasing with respect to the marginal cost c_i. Without this condition, the strategy $\beta^1(c_i)$ that we derive may fail to be strictly decreasing, thus invalidating the argument leading to it. Notice, therefore, that the derivative of $\beta^2(c_i|c_i)$ equals to

$$\begin{aligned}\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] &= -\pi^{NS}(c_i,c_i)\,\frac{(N-2)f(c_i)}{1-F(c_i)} \\ &+ \int_{c_i}^{\bar{c}} \pi_2^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\ &+ \int_{c_i}^{\bar{c}} \pi^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc\,\frac{(N-2)f(c_i)}{1-F(c_i)},\end{aligned}$$

or, after integrating the last term by parts, to

$$\begin{aligned}\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] &= \int_{c_i}^{\bar{c}} \pi_2^{NS}(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\ &+ \int_{c_i}^{\bar{c}} \pi_1^{NS}(c,c_i)\,\frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc\,\frac{(N-2)f(c_i)}{1-F(c_i)}.\end{aligned}$$

Since the derivative $\pi_1^{NS}(c_i\,|\,c^1)$ is decreasing in c_i while the derivative $\pi_2^{NS}(c_i\,|\,c^1)$ is increasing in c_i, we have

$$\begin{aligned}\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] &\geq \pi_2^{NS}(\bar{c},c_i) \\ &+ \pi_1^{NS}(c_i,c_i)\;\times\;[\,\mathbb{E}[c^1_{-i,j}\,|\,c^1_{-i,j}\geq c_i]-c_i\,]\;\frac{(N-2)f(c_i)}{1-F(c_i)},\end{aligned}$$

where $c^1_{-i,j}$ denotes the lowest value among N-2 realizations of the marginal cost c_i.

[22] Restricting attention to the game described in Lemma 1, notice that the possibility of $c_i < c^1$ does not contradict the firms' beliefs and, therefore, does not violate the consistency requirement in the definition of Nash equilibrium. We can simply assume that prior to the draw of the privately known marginal costs c_i, each firm attaches zero probability to the event $c_i < c^1$, for all i.

Hence, since $\pi_1^{NS} < 0 < \pi_2^{NS}$, if the hazard ratio $f(c_i)/[1-F(c_i)]$ is too small, then the derivative $\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,]$ can be positive.

We avoid this possibility by imposing the following condition:

Assumption 1:
The distribution of the firms' marginal costs satisfies the inequality

$$\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc \;\geq\; \sup_{c \geq c_i} \left\{ \frac{\pi_2^{NS}(c,c_i)}{-\pi_1^{NS}(c,c_i)} \right\} \frac{1-F(c_i)}{(N-2)\,f(c_i)},$$

for all $c_i \in [\underline{c}, \bar{c}]$.

In the case of uniformly distributed marginal costs, this assumption is satisfied for the Cournot oligopoly that we have described. Indeed, for $c_i \sim U[\underline{c}, \bar{c}]$, it requires that

$$\frac{\bar{c}-c_i}{N-1} \;\geq\; \frac{\bar{c}-c_i}{2(N-2)},$$

which is true for all $N \geq 3$. On the other hand, for a Bertrand oligopoly with parameter $\gamma \in [0,1)$, the assumption reduces to requiring that

$$\frac{N-2}{N-1} \;\geq\; \frac{\gamma}{2-\gamma^2},$$

which is satisfied only if the number of firms, N, is sufficiently large, relative to γ.

Assumption 1 requires that the inverse hazard rate does not decrease too rapidly. More precisely, as it is shown by the proof of the next Lemma, the inequality

$$\frac{1-F(c)}{f(c)} \;>\; \frac{\pi_2^{NS}(c,c_i)}{-\pi_1^{NS}(c,c_i)}\,\frac{1-F(c_i)}{f(c_i)}$$

remains valid for a sufficiently large interval of values $c \geq c_i$, so that the negative term in the equation of $\frac{d}{dc_i}[\beta^2(c_i|c_i)]$ dominates the positive one.

Lemma 2:
Under Assumption 1, the function $\beta^2(c_i|c_i)$ *is decreasing in* $c_i \in [\underline{c}, \bar{c}]$.

Proof: Appendix.

In the first auction, suppose that all firms follow a strictly monotone bidding strategy $b^1 = \beta^1(c)$ and consider firm i with marginal cost c_i. Then, by mimicking a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ in this auction, firm i will win the first license if and only if $\tilde{c}_i \leq c_{-i}^1$. In this case, the actual value of this license will be equal to firm i's market profit, $\pi^{NS}(c_i, c^1_{-i})$, which depends on the marginal cost c^1_{-i} of the winner of the second license.

Therefore, by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \le c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi^{NS}(c_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{\tilde{c}_i} [\,\pi^{NS}(c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1-F(c_i)]^{N-2} f(c^1)\ dc^1,\end{aligned}$$

while by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \ge c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi^{NS}(c_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{c_i} [\,\pi^{NS}(c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1-F(c_i)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{c_i}^{\tilde{c}_i} [\,\pi^{NS}(c^1, c^1) - \beta^2(c^1 \,|\, c^1)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1.\end{aligned}$$

In the second case, the extra term results from the possibility of selling the first license to a firm with marginal cost $c^1 \in [c_i, \tilde{c}_i]$. In this case, firm i bids $b^2 = \beta^2(c^1 \,|\, c^1)$ in the second auction, knowing that it has the lowest marginal cost among the remaining firms.

In both cases, the necessary first-order condition at the endpoint $\tilde{c}_i = c_i$ results in the differential equation

$$\frac{d}{d\tilde{c}_i}\{\,[1-F(c_i)]^{N-1}\,\beta^1(c_i)\} \;=\; -\,\beta^2(c_i \,|\, c_i)\ (N-1)\,[1-F(c_i)]^{N-2}\,f(c_i).$$

By solving this differential equation, along with the boundary condition

$$\beta^1(\bar{c}) = \pi^{NS}(\bar{c}, \bar{c})$$

that expresses the bidding behavior of the weakest possible type, we get the strategy

$$\beta^1(c_i) \;=\; \int_{c_i}^{\bar{c}} \beta^2(c^1 \,|\, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\ dc^1,$$

which is part of the equilibrium in our sequential auction.

Proposition 2:
In a sequential first-price auction of two oligopoly licenses, in which the winners' bids are announced at the end of each round and their marginal costs are truthfully revealed at the end of the entire auction, the following strategy profile constitutes a symmetric separating equilibrium:

– *In the first auction, each firm i bids*

$$\beta^1(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1.$$

– *In the second auction, if firm i has a marginal cost $c_i \geq c^1 = (\beta^1)^{-1}(b^1)$, where b^1 is the price at which the first license was sold, then it bids*

$$\beta^2(c_i \,|c^1) \;=\; \int_{c_i}^{\bar{c}} \pi^{NS}(c^2, c^1) \, \frac{(N-2)[1-F(c^2)]^{N-3} f(c^2)}{[1-F(c_i)]^{N-2}} \, dc^2$$

while with a marginal cost $c_i < c^1$, it bids $b^2 = \beta^2(c^1 \,|c^1)$.

Proof: Appendix.

The equation defining the bidding strategy $\beta^1(c_i)$ is a non-arbitrage condition for the winner of the first auction, which is, as it has turned out, the firm with the lowest marginal cost c_i. If this firm does not participate in the first auction, then it can win the second auction with a bid equal to $\beta^2(c^1|c^1)$, where $c^1 \geq c_i$ is the revealed lowest competing marginal cost. Therefore, to be indifferent, this firm must bid in the first auction an amount equal to its expected bid in the second round.

In the sequential first-price auction, the marginal cost of the strongest firm is known at the time of the bidding for the second license, since it is revealed after the end of the first round. Hence, in the sequential auction, unlike the case of the simultaneous auction, it is possible for both winning firms to avoid gaining a license at a price above its ex-post value.

Corollary 2:
In the sequential first-price auction, both licenses are sold at prices below their ex-post values.

Proof:
The proof for the license sold in the first auction, to the firm with the lowest marginal cost, follows from a direct argument, identical to the one for the simultaneous auction.

In addition, in the second auction, the remaining bidders know the marginal cost c^1 of the first firm and, therefore, their value for the license. As a result, they never bid an amount above that value.

□

The revelation of the marginal cost of the strongest firm after the end of the first auction makes the second license less profitable for the remaining firms. As a result, these firms' bidding for the second license becomes less aggressive.

Corollary 3:
In the sequential first-price auction, the prices at which the two licenses are sold form a super-martingale:

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,c_i] \;\leq\; \beta^1(c_i).$$

Proof:
Suppose that the first license is sold at a price $b^1 = \beta^1(c_i)$, corresponding to a marginal cost c_i. Then, conditional on this information, the expected price for the second license will be

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,b^1 = \beta^1(c_i)] \;=\; \mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,c^1_{-i} \geq c_i]$$

and, since $\beta^2(c^1|c^1) > \beta^2(c^1|c_i)$, for all $c^1 \in [c_i, \bar{c}]$,

$$\begin{aligned}\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,b^1 = \beta^1(c_i)] \;&<\; \mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c^1_{-i})\,|\,c^1_{-i} \geq c_i] \\ &=\; \beta^1(c_i),\end{aligned}$$

as required for the result.

□

The super-martingale property implies that the (ex-ante) expected price of the second license is lower than the expected price of the first license:

$$\begin{aligned}\mathbb{E}_{c_i,\,c^1_{-i}}[\,\beta^2(c^1_{-i}|c_i)\,] \;&=\; \mathbb{E}_{c_i}[\,\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i}|c_i)\,|\,b^1 = \beta^1(c_i)]\,] \\ &<\; \mathbb{E}_{c_i}[\,\beta^1(c_i)\,].\end{aligned}$$

Hence, the information revealed in the process of the sequential auction makes the expected prices decrease.

4.3 Comparison of Auction Schemes

The two auction schemes that we have examined, the simultaneous pay-your-bid auction and the sequential first-price auction, have turned out to be allocation equivalent. The licenses are allocated to the two strongest firms, that is, to the firms with the lowest marginal costs. However, the manner in which the firms bid in each scheme is different.

In the simultaneous auction, the firms submit their bids without knowing the actual value of the licenses that they try to acquire. This value is determined endogenously, by the marginal costs of the firms that will compete in the market, and is revealed only at the end of the auction. In addition, the firms cannot know whether, in case they win one of the two licenses, they will face a stronger or a weaker market competitor. Therefore, while bidding, they need to take both possibilities into account.

On the other hand, in the sequential auction, the firms bidding for the second license know its actual value, since the marginal cost of the first oligopolist has been revealed by the winning bid in the first auction. Furthermore, in the auction for the first license, the firms know that if they win, then they will face a weaker market competitor. Therefore, they can bid more aggressively, since they are protected from the more negative of the two possibilities.

The following result shows that these informational differences do not affect the revenue generated by the auctioneer in the two schemes.

Proposition 3:
The simultaneous pay-your-bid auction and the sequential first-price auction of two Cournot oligopoly licenses result in the same expected revenue for the auctioneer.

Proof:
It is easy to show, by changing the order of integration in the definition of $\beta(c_i)$, that

$$\begin{aligned}\mathbb{P}[c^2_{-i} \geq c_i]\;\beta(c_i) &= [1-F(c_i)]^{N-1}\;\beta^1(c_i) \\ &+ (N-1)[1-F(c_i)]^{N-2}F(c_i)\int_{\underline{c}}^{c_i}\beta^2(c_i|c^1)\,\frac{f(c^1)}{F(c_i)}\,dc^1.\end{aligned}$$

This means that the expected payments of a firm with marginal cost c_i in the simultaneous auction, $R^D_{NS}(c_i)$, and in the sequential auction, $R^S_{NS}(c_i)$, are equal.

Therefore, since this is true for any $c_i \in [\underline{c}, \bar{c}]$, it follows that

$$N\int_{\underline{c}}^{\bar{c}} R^D_{NS}(c_i)f(c_i)\,dc_i \;=\; N\int_{\underline{c}}^{\bar{c}} R^S_{NS}(c_i)f(c_i)\,dc_i,$$

so that the two auction schemes raise the same expected revenue.

□

Hence, the auctioneer is indifferent, with respect to the revenue he expects to raise, between the two auction schemes. Similarly, the bidders are indifferent, with respect to the payments they expect to make, between the simultaneous and the sequential auction. The two auction formats, however, allocate each of the two licenses at a different price.

Proposition 4:
The stronger of the two oligopolists pays a higher price for his license in the sequential first-price auction than in the simultaneous pay-your-bid auction; the weaker oligopolist pays a lower price for his license in the sequential auction than in the simultaneous auction:

$$\beta^1(c_i) \;>\; \beta(c_i) \;>\; \mathbb{E}_{c^1_{-i}}[\,\beta^2(c_i\,|\,c^1_{-i})\,|\,c^1_{-i} < c_i].$$

Proof: Appendix.

Therefore, in the first auction of the sequential format, the firms bid more aggressively than in the simultaneous auction, knowing that if they win, they will necessarily face a weaker market competitor. On the other hand, the firms participating in the second auction bid less aggressively, on average, since they know that they will have to face a stronger market competitor.

Corollary 4:
The stronger of the two oligopolists makes a higher total profit in the simultaneous pay-your-bid auction. The weaker oligopolist makes a higher total profit in the sequential first-price auction.

Proof:
Since both auction formats result to the same market supply and profits, any change in the firms' total profits will be the consequence of a change in the prices that the firms pay for their licenses. Therefore, the result follows directly from Proposition 4.

□

Hence, an auctioneer aiming at a more equal distribution of the wealth generated in the oligopolistic market will prefer the sequential first-price auction to the simultaneous pay-your-bid auction.

5 Positive Signaling: Cournot Competition

When signaling is possible, we need to revise the firms' valuations for the oligopoly licenses so as to incorporate to them the informational rents that the firms can extract. In the case of Cournot competition, in which the signaling incentives are positive, the firms' valuations shall be adjusted upwards.

To demonstrate the need for this adjustment, consider an auction of a single license to compete against a monopolist with known marginal cost c^1. When signaling is not possible, a firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$ would be willing to bid for the license an amount up to

$$\pi^{NS}(c_i, c^1) \;=\; \pi(c_i \,|\, c_i, c^1).$$

If signaling becomes possible, then, by mimicking a marginally stronger type $\tilde{c}_i < c_i$, firm i can increase its market profit, in case it wins the license, by approximately

$$-\pi_2(c_i \,|\, \tilde{c}_i, c^1)\, d\tilde{c}_i \;>\; 0.$$

Therefore, the maximal amount that the firm would be willing to bid exceeds $\pi^{NS}(c_i, c^1)$.

Since the effects of mimicking a different type depend on the auction format and on the equilibrium strategies that the bidders use, the manner in which the firms adjust their valuations will also depend on these elements. Therefore, the firms' valuations will be different in each auction environment that we consider.

Overall, under positive signaling, a firm's deviation to signaling a stronger type will have two effects. First, assuming that the bidding strategies are monotone, it will increase the probability of acquiring a license. Second, it will increase the profitability of the license that the firm may win. Hence, to offset both these effects, the firms must bid more aggressively than they would do if signaling were not possible.

5.1 Simultaneous Auction

Suppose that all firms follow a strictly decreasing bidding strategy $b = \beta(c)$ and consider firm i with marginal cost c_i. If firm i mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$ during the auction, by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$, then its expected total payoff will be

$$\begin{aligned} \Pi(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(c^1)\, dc^1 \\ &+\; \int_{\tilde{c}_i}^{\bar{c}} \pi(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\ &-\; \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\; \beta(\tilde{c}_i). \end{aligned}$$

The first-order condition with respect to $\tilde{c}_i$ results in the equation

$$\begin{aligned}
\frac{d}{d\tilde{c}_i}\{\mathbb{P}[c^2_{-i} \geq c_i]\,\beta(c_i)\} \;&=\; \int_{\underline{c}}^{c_i} \pi_2(c_i\,|\,c_i, c^1)\;(N-1)\,[1-F(c_i)]^{N-2}\,f(c^1)\;dc^1 \\
&+\; \int_{c_i}^{\bar{c}} \pi_2(c_i\,|\,c_i, c^1)\;(N-1)\,[1-F(c^1)]^{N-2}\,f(c^1)\;dc^1 \\
&-\; \int_{\underline{c}}^{c_i} \pi(c_i\,|\,c_i, c^1)\,(N-1)(N-2)\,[1-F(c_i)]^{N-3}\,f(c_i)f(c^1)\,dc^1,
\end{aligned}$$

which requires that any increase in the firm's expected market profit from a deviation to $\tilde{c}_i \neq c_i$, as this may be augmented by the expected gains from false signaling[23], must be offset by an increase in the firm's expected payment in the auction.

The differential equation derived from the first-order condition, along with the boundary condition expressing the behavior of the weakest type, $c_i = \bar{c}$,

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi(\bar{c}\,|\,\bar{c}, c)\,f(c)\,dc,$$

provides the equilibrium strategy for this setting.

Proposition 5:
In the simultaneous pay-your-bid auction of two Cournot oligopoly licenses, in which the winners' bids are revealed at the end of the auction, there is a symmetric separating equilibrium given by the strategy

$$\begin{aligned}
\beta(c_i) \;&=\; \int_{c_i}^{\bar{c}}\int_{\underline{c}}^{c^2} \pi(c^2\,|\,c^2, c^1)\;\frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\,f(c^2)\,f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\;dc^1\,dc^2 \\
&+\; \int_{c_i}^{\bar{c}}\int_{\underline{c}}^{c^2} -\pi_2(c^2\,|\,c^2, c^1)\;\frac{(N-1)\,[1-F(c^2)]^{N-2}\,f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\;dc^1\,dc^2 \\
&+\; \int_{c_i}^{\bar{c}}\int_{c^2}^{\bar{c}} -\pi_2(c^2\,|\,c^2, c^1)\;\frac{(N-1)\,[1-F(c^1)]^{N-2}\,f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\;dc^1\,dc^2.
\end{aligned}$$

In this equilibrium, the firm with the lowest marginal cost gains its license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain its license at a price above its ex-post value.

[23] In particular, when $c_i < c^1_{-i} < c^2_{-i}$, the change in the firm's expected market profit is entirely the consequence of false signaling.

Proof:
Notice that the strategy $\beta(c_i)$ can be expressed as

$$\beta(c_i) \;=\; \int_{c_i}^{\bar{c}} u(c^2)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\,F(c^2)\,f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^2,$$

where

$$\begin{aligned} u(c) \;=\; & \int_{\underline{c}}^{c} \pi(c\,|\,c, c^1) \times \frac{f(c^1)}{F(c)}\, dc^1 \\ &+ \int_{\underline{c}}^{c} [\,-\pi_2(c\,|\,c, c^1)\,\frac{1-F(c)}{(N-2)f(c)}\,] \times \frac{f(c^1)}{F(c)}\, dc^1 \\ &+ \int_{c}^{\bar{c}} [\,-\pi_2(c\,|\,c, c^1)\,\frac{1-F(c)}{(N-2)f(c)}\,] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c)]^{N-2} F(c)}\, dc^1 \end{aligned}$$

for $c \in [\underline{c}, \bar{c}]$, is the valuation of a firm with marginal cost c, assuming that its market opponent is stronger and taking into account the informational rents from false signaling.

Therefore, the proof of this result parallels the one of Proposition 2, with $u(c)$ in place of $v^{NS}(c)$. Its details, in particular, the argument establishing that $u(c)$ is decreasing, can be found in the Appendix.

□

The first term in the bidding strategy,

$$\beta_{NS}(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi(c^2\,|\,c^2, c^1)\, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\,f(c^1)f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2,$$

corresponds to the amount that a firm with marginal cost c_i would bid, if signaling were not possible.

The second and third terms,[24]

$$\begin{aligned} \beta_S(c_i) \;=\; & \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} -\pi_2(c^2\,|\,c^2, c^1)\, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2 \\ &+ \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} -\pi_2(c^2\,|\,c^2, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]}\, dc^1\, dc^2, \end{aligned}$$

[24] The two double integrals do not allow, of course, for the possibility of $c^1_{-i} > c^2_{-i}$. Rather, in each case, the outer integral determines a value $c^2 \in [c_i, \bar{c}]$ such that $c^2 \leq c^2_{-i}$, while $c^1_{-i} \leq c^2_{-i}$. This creates two possibilities, namely, either $c^1_{-i} \leq c^2 \leq c^2_{-i}$, corresponding to the first signaling term, or $c^2 \leq c^1_{-i} \leq c^2_{-i}$, corresponding to the second signaling term.

correspond to the amount by which the firms should augment their bids so as to offset possible gains from false signaling by the other firms. Even though there can be no false signaling in equilibrium, without this amount, it would be possible for a firm to deviate into mimicking a stronger type and, therefore, to increase both the probability of winning a license and, through false signaling, the value of that license.

5.2 Sequential Auction

In the sequential auction, the winning bid in the first round reveals the marginal cost c^1 of the strongest oligopolist. Therefore, similarly to the case in which signaling is not possible, in the second round, the firms know precisely the value of the license for which they bid. In addition, they know that the other firms' marginal costs are bounded below by c^1; thus, they update their beliefs, so that

$$c_i \sim \tilde{F}(c) = \frac{F(c) - F(c^1)}{1 - F(c^1)}.$$

Since the privately known first-period bids do not affect the firms' incentives in the second auction, our second-period bidding environment belongs to the class of auctions studied by Das Varma [9], Goeree [14] and Katzman and Rhodes-Kropf [31]. In the following Lemma, we apply their analysis to our setting:

Lemma 3:
Suppose that $N-1$ firms, whose marginal costs are i.i.d. according to the distribution function $F(.)$ on $[\underline{c}, \bar{c}]$, compete in a first-price auction for a license to participate in a Cournot oligopoly against a firm with known marginal cost c^1. In addition, suppose that the firms believe that the unknown marginal costs are bounded below by the value $c^1 \in [\underline{c}, \bar{c}]$. Then, assuming that the winner's bid is revealed at the end of the auction, the following strategy constitutes a symmetric equilibrium:

For a marginal cost $c_i \geq c^1$, firm i bids

$$\beta^2(c_i\,|c^1) \;\; = \;\; \int_{c_i}^{\bar{c}} \left[\pi(c|c,c^1) - \pi_2(c|c,c^1)\,\frac{1-F(c)}{(N-2)f(c)}\right] \frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc,$$

while for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1\,|c^1)$.

In addition, for any $c^1 \in [\underline{c}, \bar{c}]$, the strategy $\beta^2(c_i|c^1)$ is strictly decreasing in $c_i \in (c^1, \bar{c}\,]$, so that, along the equilibrium path, an auction price $b^2 < \beta^2(c^1|c^1)$ fully reveals the marginal cost of the winning firm.

Proof: Appendix.

Every firm i submits a bid that is equal to the value that its strongest competitor is expected to have for the license, assuming that this competitor has marginal cost $c \geq c_i$,

$$\beta^2_{NS}(c_i \,|\, c^1) \;=\; \int_{c_i}^{\bar{c}} \pi(c \,|\, c, c^1) \, \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}} \, dc,$$

augmented by the amount needed to offset possible gains from false signaling by its competitors, namely,

$$\beta^2_{S}(c_i \,|\, c^1) \;=\; \int_{c_i}^{\bar{c}} -\pi_2(c \,|\, c, c^1) \, \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}} \, dc.$$

Without this amount, it would be possible for a firm i to deviate into mimicking $\tilde{c}_i < c_i$ and to increase both the probability of winning a license and, through false signaling, the value of that license. For the second gain to be offset, each firm needs to bid above $\beta^2_{NS}(c_i \,|c^1)$, by an amount at least as large as $\beta^2_{S}(c_i \,|c^1)$.

To ensure that the strategy $\beta^2(c_i|c_i)$ is decreasing with respect to the marginal cost c_i, we will need to modify Assumption 1 in the following manner:

Assumption 2:
The distribution of the firms' marginal costs satisfies the inequality

$$\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}} \, dc \;\geq\; \sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\} \, \frac{1-F(c_i)}{(N-2)\, f(c_i)},$$

for all $c_i \in [\underline{c}, \bar{c}]$, where

$$\tilde{v}_1(c, c^1) \;=\; \frac{d}{dc}[\pi(c \,|\, c, c^1)] \;-\; \frac{d}{dc}[\pi_2(c \,|\, c, c^1)] \, \frac{1-F(c)}{(N-2)\, f(c)}.$$

For the Cournot duopoly that we have described,

$$\sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\} \;=\; \sup_{c \geq c_i} \left\{ \frac{\pi_1(c \,|\, c, c_i)}{-\pi_2(c \,|\, c, c_i)} \right\},$$

so that Assumption 2 reduces to Assumption 1. In particular, for marginal costs $c_i \sim U[\underline{c}, \bar{c}]$, the assumption is always satisfied.

Lemma 4:
Under Assumption 2, the function $\beta^2(c_i|c_i)$ is decreasing in $c_i \in [\underline{c}, \bar{c}]$.

Proof: Appendix.

In the first auction, arguing in the same manner as in the non-signaling case, suppose that all firms follow a strictly monotone bidding strategy $\beta^1(c)$ and consider firm i with marginal cost c_i. By bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \leq c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi(c_i \,|\, \tilde{c}_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{\tilde{c}_i} [\,\pi(c_i \,|\, c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1-F(c_i)]^{N-2} f(c^1)\ dc^1,\end{aligned}$$

while by bidding $\tilde{b}_i = \beta(\tilde{c}_i)$ for $\tilde{c}_i \geq c_i$, firm i expects a total payoff

$$\begin{aligned}\Pi(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} [\,\pi(c_i \,|\, \tilde{c}_i, c^1) - \beta^1(\tilde{c}_i)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{\underline{c}}^{c_i} [\,\pi(c_i \,|\, c_i, c^1) - \beta^2(c_i \,|\, c^1)\,]\ (N-1)[1-F(c_i)]^{N-2} f(c^1)\ dc^1 \\ &+ \int_{c_i}^{\tilde{c}_i} [\,\pi(c_i \,|\, c^1, c^1) - \beta^2(c^1 \,|\, c^1)\,]\ (N-1)[1-F(c^1)]^{N-2} f(c^1)\ dc^1.\end{aligned}$$

In both cases, the necessary first-order condition at the endpoint $\tilde{c}_i = c_i$ results in the differential equation

$$\begin{aligned}\frac{d}{d\tilde{c}_i}\{\,[1-F(c_i)]^{N-1}\,\beta^1(c_i)\} &= \int_{c_i}^{\bar{c}} \pi_2(c_i \,|\, c_i, c^1)\ (N-1)\,[1-F(c^1)]^{N-2}\,f(c^1)\ dc^1 \\ &- \beta^2(c_i \,|\, c_i)\ (N-1)\,[1-F(c_i)]^{N-2}\,f(c_i).\end{aligned}$$

By solving this differential equation, along with the boundary condition

$$\beta^1(\bar{c}) = \pi(\bar{c} \,|\, \bar{c}, \bar{c}),$$

we get the strategy

$$\begin{aligned}\beta^1(c_i) &= \int_{c_i}^{\bar{c}} \beta^2(c^1 \,|\, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\ dc^1 \\ &- \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} \pi_2(c^2 \,|\, c^2, c^1)\ \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\ dc^1\, dc^2,\end{aligned}$$

for the equilibrium of the sequential auction.

Proposition 6:
In a sequential first-price auction of two Cournot oligopoly licenses, in which the winners' bids are revealed at the end of each auction, the following strategy profile constitutes a symmetric separating equilibrium:

– *In the first auction, each firm i bids*

$$\begin{aligned}
\beta^1(c_i) &= \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1) \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \\
&+ \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \\
&+ \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 .
\end{aligned}$$

– *In the second auction, if firm i has a marginal cost $c_i \geq c^1 = (\beta^1)^{-1}(b^1)$, where b^1 is the price at which the first license was sold, then it bids*

$$\begin{aligned}
\beta^2(c_i \,|\, c^1) &= \int_{c_i}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1) \frac{(N-2)[1-F(c^2)]^{N-3} f(c^2)}{[1-F(c_i)]^{N-2}} \, dc^2 \\
&+ \int_{c_i}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \frac{[1-F(c^2)]^{N-2}}{[1-F(c_i)]^{N-2}} \, dc^2 ,
\end{aligned}$$

while with a marginal cost $c_i < c^1$, it bids $b^2 = \beta^2(c^1 \,|c^1)$.

Proof:
Notice that the strategy $\beta^1(c_i)$ can be expressed as

$$\beta(c_i) = \int_{c_i}^{\bar{c}} v(c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} \, f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^1 ,$$

where

$$\begin{aligned}
v(c) &= \int_{c}^{\bar{c}} \pi(c \,|\, c, c^1) \times \frac{(N-2) \, [1-F(c^1)]^{N-3} f(c^1)}{[1-F(c)]^{N-2}} \, dc^1 \\
&+ \int_{c}^{\bar{c}} [-\pi_2(c \,|\, c, c^1) \, \frac{1-F(c)}{(N-2) f(c)} \,] \times \frac{(N-2) \, [1-F(c^1)]^{N-3} f(c^1)}{[1-F(c)]^{N-2}} \, dc^1 \\
&+ \int_{c}^{\bar{c}} [-\pi_2(c \,|\, c, c^1) \, \frac{1-F(c)}{(N-2) f(c)} \,] \times \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c)]^{N-1}} \, dc^1 ,
\end{aligned}$$

for $c \in [\underline{c}, \bar{c}]$, is the valuation of a firm with marginal cost c, assuming that its market opponent is weaker.

Therefore, the proof of this result parallels the one of Proposition 5, with $v(c)$ in place of $u(c)$. Its details can be found in the Appendix.

□

Similarly to the case of non-signaling, the equation defining the bidding strategy $\beta^1(c_i)$ is a non-arbitrage condition for the firm with the lowest marginal cost, c_i. For this firm to be indifferent between winning the first or the second auction, its bid in the first round must exceed its expected bid in the second round by precisely its expected gain from signaling a stronger type.

The first term of the bidding strategy for the first auction,

$$\beta^1_{NS}(c_i) \;=\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^1) f(c^2)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1,$$

corresponds, again, to the amount that firm i would bid if signaling were not possible. The remaining two terms,

$$\begin{aligned} \beta^1_S(c_i) \;&=\; \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \\ &+\; \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1, \end{aligned}$$

correspond to the amounts that the firms must add to their bids in order to offset possible gains from false signaling by their competitors.

Because of the information revealed in the first round of the sequential auction, the weaker of the two oligopolists is able to avoid the possibility of winning his license at a price above its ex-post value. Thus, along the equilibrium path, both oligopolists make a positive profit.

In addition, the revelation of the marginal cost of the strongest firm makes the second license less profitable and, therefore, the firms' bidding for it less aggressive. As a result, the prices of the two licenses form a super-martingale,

$$\mathbb{E}_{c^1_{-i}}[\beta^2(c^1_{-i} | \, c_i) \,|\, c_i] \;\leq\; \beta^1(c_i),$$

so that the expected price of the second license is lower than that of the first license.

5.3 Comparison of Auction Schemes

Our analysis of the simultaneous and the sequential auctions under positive signaling parallels the analysis of the same auctions without signaling. Non-surprisingly, so do the results regarding the comparison of the equilibria that we derived.

Proposition 7:
The simultaneous pay-your-bid auction and the sequential first-price auction of two Cournot oligopoly licenses result in the same expected revenue for the auctioneer.

Therefore, the informational differences between the two auction schemes do not affect the expected revenue of the auctioneer or the expected payment of the bidders.

Proposition 8:
The stronger of the two oligopolists pays a higher price for his license in the sequential first-price auction than in the simultaneous pay-your-bid auction; the weaker oligopolist pays a lower price for his license in the sequential auction than in the simultaneous auction:

$$\beta^1(c_i) \;>\; \beta(c_i) \;>\; \mathbb{E}_{c^1_{-i}}[\,\beta^2(c_i \,|\, c^1_{-i}) \,|\, c^1_{-i} < c_i].$$

Therefore, the stronger of the two oligopolists makes a higher total profit in the simultaneous pay-your-bid auction while the weaker oligopolist makes a higher total profit in the sequential first-price auction.

Proof: Appendix.

Hence, an auctioneer aiming at a more equal distribution of the wealth generated in the Cournot duopoly will still prefer the sequential auction to the simultaneous one, even when signaling is possible.

6 Negative Signaling: Bertrand Competition

In the case of Bertrand competition, the firms have an incentive to signal a weaker type. Therefore, opposite to the case of Cournot competition, the firms' valuations shall be adjusted downwards. Because of this adjustment, if the firms' signaling incentive is too strong, it is possible that a positive measure of bidder types will have valuations below zero. To avoid this problem, we need to assume the presence of a large number of firms competing in the auction.

Under this assumption, we can construct an equilibrium in strictly monotone bidding strategies for the simultaneous auction. On the other hand, in the sequential auction, since it is not possible to balance the bidders' signaling profits from deviating into waiting for the second round, such an equilibrium turns out not to exist.

6.1 Simultaneous Auction

By repeating the argument that we used for the Cournot oligopoly, that is, by assuming the use of a strictly decreasing bidding strategy $b = \beta(c)$ and considering the necessary first-order condition for the expected payoff function $\Pi(\tilde{c}_i|c_i)$ of some firm i at $\tilde{c}_i = c_i$, we can derive the equilibrium for this setting.

Proposition 9:
In the simultaneous pay-your-bid auction of two Bertrand oligopoly licenses, with the winners' bids revealed at the end of the auction, if there are sufficiently many bidders, then there is a symmetric separating equilibrium given by the strategy

$$\begin{aligned}
\beta(c_i) &= \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} \pi(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, f(c^2)\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2 \\
&+ \int_{c_i}^{\bar{c}} \int_{\underline{c}}^{c^2} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)\,[1-F(c^2)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2 \\
&+ \int_{c_i}^{\bar{c}} \int_{c^2}^{\bar{c}} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)\,[1-F(c^1)]^{N-2}\, f(c^1)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^1 \, dc^2.
\end{aligned}$$

In this equilibrium, the firm with the lowest marginal cost gains its license at a price below its ex-post value. The firm with the second-lowest marginal cost, however, may gain its license at a price above its ex-post value.

Proof: Appendix.

Notice that by understating its strength, a firm gains in terms of its expected market profit and of a lower payment in the auction, assuming that it wins an oligopoly license. On the other hand, it suffers the cost of a lower probability of winning the auction. This cost increases as the number of the bidders in the auction, N, becomes larger. Therefore, if N is sufficiently large, the cost is so severe that it can always counter-balance possible gains from false signaling.

6.2 Sequential Auction

In the sequential auction, the firms' incentive to signal a weaker type turns out to be too strong. Contrary to the case of the simultaneous auction, it is not possible to construct a symmetric separating equilibrium.

Proposition 10:
In a sequential first-price auction of two Bertrand oligopoly licenses, in which the winners' bids are revealed at the end of each auction, there is no symmetric equilibrium in monotone strategies.

Proof: Appendix.

In the presence of a sufficiently large number of bidders, as shown in Das Varma [9], the strategy $\beta^2(c_i|c^1)$, given in Lemma 3, forms the unique symmetric equilibrium for the continuation game that follows the allocation of the first license to a firm with marginal cost $c^1 \in [\underline{c}, \bar{c}]$. In addition, by adapting Assumption 2 to the Bertrand oligopoly setting, one can show that $\beta^2(c|c)$ is decreasing in c. Finally, by replicating the argument leading to Proposition 6, one can derive the bidding strategy $\beta^1(c_i)$, identical to the one used in the Cournot oligopoly, as the unique solution to the necessary first-order condition.

This strategy, however, cannot be part of an equilibrium. Although, the non-signaling component of $\beta^2(c_i|c^1)$ is sufficiently more aggressive than the non-signaling component of $\beta^1(c_i)$, so that to just eliminate the incentive to wait for the second round (if signaling were not possible), the signaling component of $\beta^2(c_i|c^1)$ cannot counter-balance the corresponding component of $\beta^1(c_i)$. As a result, each firm has a profitable deviation from β^1 into waiting for the second round.

In particular, trying to diminish the potential gains from signaling by increasing the number of firms, as in the case of the simultaneous auction, cannot produce any result. The deviation into waiting for the second round does not cost any firm in terms of the probability of acquiring a license, so, changing the number of firms is ineffective.

7 Conclusion

We have examined two multi-unit auction schemes with allocative and, possibly, informational externalities, in particular, two auctions of oligopoly licenses.

When there is no signaling, we have provided a rationale for the use of a sequential procedure. The information released during this procedure leads to more informative bidding. Even though this does not affect the seller's expected revenue, or the bidders' expected payments, the two winners are protected from the possibility of regret, that is, from buying a license at a price that exceeds its ex-post value. In addition, the strongest oligopolist has to pay a higher price for his license than he would pay in a simultaneous auction, whereas the weaker oligopolist pays a lower price. Therefore, the sequential auction results in a more even distribution of the wealth generated in the oligopoly.

When signaling is possible, these results remain valid only in the case of positive signaling incentives, as in the Cournot oligopoly. On the other hand, with negative signaling incentives, as in the Bertrand oligopoly, there is no symmetric monotone equilibrium for the sequential auction. Hence, in this environment, an efficient allocation is achieved only by means of a simultaneous auction.

The two auction formats will cease to be revenue equivalent, if we consider affiliated marginal costs. In this case, according to the intuition of the linkage principle, the sequential format will dominate, in terms of revenue, the simultaneous auction. The two auctions will also generate different expected seller revenues, if the firms face participation costs. In particular, if the winning bid in the first round of the sequential auction is sufficiently low, then some bidder types that would otherwise not participate may decide to bid in the second round. In this case, however, the auction scheme that is preferable for the seller may depend on the distribution of the firms' marginal costs.

A seller may also increase his expected revenue by adopting different information revelation rules and, therefore, allowing for different signaling possibilities. According to the intuition derived from the study of the auction of a single license, schemes that reveal more information about the winners will be revenue dominant in the case of positive signaling incentives, while schemes that disable signaling will be dominant in the case of negative signaling incentives.

Finally, it would be interesting to investigate experimentally the bidding behavior in a sequential auction with negative informational externalities. An experimental study may reveal patterns of behavior that can be of interest to sellers that would like to consider the use of a sequential auction scheme.

These extensions are the subject of future research.

Appendix: Proof of Results

Proof of Proposition 1:
It is straightforward to verify that the function $\beta(c_i)$ is a solution to the differential equation that resulted from the necessary first-order condition. In addition, by using L'Hospital's rule, it is easy to check that

$$\lim_{c_i \to \bar{c}} \beta(c_i) = \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c} \,|\, \bar{c}, c)\, f(c)\, dc,$$

as required by the boundary condition.

Since the equation that produced the strategy $\beta(c_i)$ was only a necessary condition, we still need to establish that it is optimal for any bidder i with marginal cost c_i to bid $b_i = \beta(c_i)$, if all other bidders follow this bidding strategy.

Suppose that firm i bids $\tilde{b}_i = \beta(\tilde{c}_i)$, for $\tilde{c}_i \in [\underline{c}, \bar{c}]$ while having a marginal cost c_i. Then, by changing its bid marginally, that is, by mimicking a marginally different type, it can change its expected payoff by

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \quad &= \quad -\frac{d}{d\tilde{c}_i}\{\, \mathbb{P}[c_{-i}^2 \geq \tilde{c}_i]\, \beta(\tilde{c}_i)\,\} \\ &\quad - \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\ (N-1)(N-2)\, [1 - F(\tilde{c}_i)]^{N-3}\, f(\tilde{c}_i) f(c^1)\ dc^1 \end{aligned}$$

Substituting the expression for $\beta(c_i)$ results in

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \quad &= \quad \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(\tilde{c}_i, c^1)\ (N-1)(N-2)[1 - F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1 \\ &\quad - \int_{\underline{c}}^{\tilde{c}_i} \pi^{NS}(c_i, c^1)\ (N-1)(N-2)[1 - F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1 \end{aligned}$$

Since the function $\pi^{NS}(c_i, c^1)$ is decreasing in the marginal cost c_i, the change in the firm's expected payoff is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

showing that the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ attains its maximum at $\tilde{c}_i = c_i$.

To show that the strategy $\beta(c_i)$ is decreasing, we can calculate its derivative to be

$$\frac{d\beta}{dc_i}(c_i) \;=\; \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \times \left[-v(c_i) + \int_{c_i}^{\bar{c}} v(c^2) \frac{-\frac{d}{dc^2}\mathbb{P}[c_{-i}^2 \geq c^2]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \, dc^2 \right],$$

where

$$v^{NS}(c) \;=\; \int_{\underline{c}}^{c} \pi^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1$$

for $c \in [c_i, \bar{c}]$, is the expected market profit of the strongest non-winning firm. Therefore, if the function $v(c)$ is decreasing, we can conclude that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \times [\, -v(c_i) + v(c_i) \,] \;=\; 0,$$

as required for the strategy $\beta(c_i)$ to be decreasing.

By differentiating the function $v^{NS}(c)$, we get

$$\begin{aligned}
\frac{dv^{NS}}{dc}(c) \;&=\; \pi^{NS}(c, c) \, \frac{f(c)}{F(c)} \\
&+\; \int_{\underline{c}}^{c} \pi_1^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1 \\
&-\; \int_{\underline{c}}^{c} \pi^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1 \, \frac{f(c)}{F(c)},
\end{aligned}$$

and, after integrating the last term by parts,

$$\begin{aligned}
\frac{dv^{NS}}{dc}(c) \;&=\; \int_{\underline{c}}^{c} \pi_1^{NS}(c, c^1) \, \frac{f(c^1)}{F(c)} \, dc^1 \\
&+\; \int_{\underline{c}}^{c} \pi_2^{NS}(c, c^1) \, \frac{F(c^1)}{F(c)} \, dc^1 \, \frac{f(c)}{F(c)}.
\end{aligned}$$

Since the firms' marginal costs are distributed in a logconcave manner, the expression $f(c)/F(c)$ is decreasing. Therefore,

$$\frac{dv^{NS}}{dc}(c) \;\leq\; \int_{\underline{c}}^{c} [\, \pi_1^{NS}(c, c^1) + \pi_2^{NS}(c, c^1) \,] \, \frac{f(c^1)}{F(c)} \, dc^1,$$

and, since $\pi_1^{NS} + \pi_2^{NS} < 0$, we conclude that

$$\frac{dv^{NS}}{dc}(c) < 0,$$

completing the argument.

$\square$

Proof of Corollary 1:
Suppose that firm i has the lowest marginal cost among all firms, namely, $c_i \in [\underline{c}, \bar{c}]$. Then, in equilibrium, it will win one of the two licenses, at a price $\beta(c_i)$, for a market profit $\pi^{NS}(c_i, c^1)$, where $c^1 \geq c_i$ is the lowest competing marginal cost. Its overall payoff, therefore, will be $\pi^{NS}(c_i, c^1) - \beta(c_i)$. Since the function $\pi^{NS}(c_i, c^1)$ is increasing in c^1,

$$\pi^{NS}(c_i, c^1) - \beta(c_i) \;\geq\; \pi^{NS}(c_i, c_i) - \beta(c_i),$$

so, it suffices to show that

$$\mathbb{P}[c_{-i}^2 \geq c_i]\;\{\pi^{NS}(c_i, c_i) - \beta(c_i)\} \;\geq\; 0.$$

Notice that this last inequality is true for the boundary value $c_i = \bar{c}$. Furthermore,

$$\begin{aligned}
\frac{d}{dc_i}\left\{\mathbb{P}[c_{-i}^2 \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\right\} \;=&\\
&- (N-1)(N-2)[1 - F(c_i)]^{N-3} F(c_i) f(c_i)\; \pi^{NS}(c_i, c_i)\\
&+ \left\{[1 - F(c_i)]^{N-1} + (N-1)[1 - F(c_i)]^{N-2} F(c_i)\right\} \frac{d}{dc_i}[\pi^{NS}(c_i, c_i)]\\
&+ \int_{\underline{c}}^{c_i} \pi^{NS}(c_i, c^1)\; (N-1)(N-2)\,[1 - F(c_i)]^{N-3} f(c_i) f(c^1)\; dc^1
\end{aligned}$$

By integrating the last term by parts, this derivative becomes

$$\begin{aligned}
\frac{d}{dc_i}\left\{\mathbb{P}[c_{-i}^2 \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\right\} \;=&\\
&\left\{[1 - F(c_i)]^{N-1} + (N-1)[1 - F(c_i)]^{N-2} F(c_i)\right\} \frac{d}{dc_i}[\pi^{NS}(c_i, c_i)]\\
&- \int_{\underline{c}}^{c_i} \pi_2^{NS}(c_i, c^1)\; (N-1)(N-2)\,[1 - F(c_i)]^{N-3}\, F(c_i)\, f(c^1)\; dc^1.
\end{aligned}$$

Since both terms are negative, it follows that

$$\frac{d}{dc_i}\left\{\mathbb{P}[c^2_{-i} \geq c_i]\,[\pi^{NS}(c_i, c_i) - \beta(c_i)]\right\} \;\leq\; 0,$$

proving firm i's realized payoff to be always positive.

To show that the firm with the second-lowest marginal cost may win a license at a price above its ex-post value, consider a firm with marginal cost $c_i = \bar{c}$. In equilibrium, such a firm will bid

$$\beta(\bar{c}) \;=\; \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

the expected value of the license, given that $N-2$ firms have marginal cost equal to $\bar{c}$.

If the firm wins the license, then, in the market, it may face an opponent with marginal cost $\underline{c}$. In this case, it will make a market profit $\pi^{NS}(\bar{c}, \underline{c}) \;\leq \pi^{NS}(\bar{c}, c)$, for all $c \in [\underline{c}, \bar{c}]$. Therefore,

$$\pi^{NS}(\bar{c}, \underline{c}) < \int_{\underline{c}}^{\bar{c}} \pi^{NS}(\bar{c}, c)\, f(c)\, dc,$$

for a negative overall profit.

□

Proof of Lemma 2:
The derivative of the function $\beta^2(c_i|c_i)$ with respect to the variable c_i is equal to

$$\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;\;=\;\; \int_{c_i}^{\bar{c}} -\pi_1^{NS}(c, c_i)\; w(c, c_i)\; \frac{(N-2)[1-F(c)]^{N-3} f(c)}{[1-F(c_i)]^{N-2}}\, dc,$$

where

$$w(c, c_i) \;\;=\;\; \frac{\pi_2^{NS}(c, c_i)}{-\pi_1^{NS}(c, c_i)} \;-\; \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]}.$$

The expression $w(c, c_i)$ is negative for $c = c_i$, positive for $c = \bar{c}$, continuous and increasing with respect to $c \in [c_i, \bar{c}]$. Hence, there exists a value $c^* = c^*(c_i) \in (c_i, \bar{c})$ such that

$$w(c, c_i)\;\begin{cases} < 0, & \text{for } c \in [c_i, c^*); \\ = 0, & \text{for } c = c^*; \\ > 0, & \text{for } c \in (c^*, \bar{c}\,]. \end{cases}$$

It follows that

$$\frac{d}{dc_i}[\,\beta^2(c_i|c_i)\,] \;<\; -\pi_1^{NS}(c^*,c_i) \int_{c_i}^{\bar{c}} w(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc$$

and, since Assumption 1 implies that

$$\int_{c_i}^{\bar{c}} w(c,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \;\leq\; 0,$$

we can conclude that $\beta^2(c_i|c_i)$ is decreasing.

□

Proof of Proposition 2:
Suppose that all firms follow the bidding strategy (β^1, β^2) and consider firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$. The optimality of bidding $\beta^2(c_i\,|\,c^1)$ in the second auction, following the sale of the first license at a price b^1 corresponding to a marginal cost $c^1 = (\beta^1)^{-1}(b^1)$, has been established in Lemma 1.[25] Therefore, we only need to examine the optimality of bidding $\beta^1(c_i)$ in the first auction.

Obviously, firm i cannot gain from submitting a bid above $\beta^1(\underline{c})$ or below $\beta^1(\bar{c})$. So, suppose that it mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$, that is, it bids $\beta^1(\tilde{c}_i)$. If $\tilde{c}_i \leq c_i$, then, by changing its bid marginally, firm i will change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i\,|\,c_i) \;&=\; \pi^{NS}(c_i,\tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i) \\
&-\; \pi^{NS}(c_i,\tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2}f(\tilde{c}_i) \\
&-\; \beta^2(c_i|\tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2}f(\tilde{c}_i) \\
&-\; \frac{d}{d\tilde{c}_i}\{\,[1-F(\tilde{c}_i)]^{N-1}\,\beta^1(\tilde{c}_i)\,\}.
\end{aligned}$$

After substituting the appropriate expression for the last term, the change in the expected payoff of firm i becomes

[25] In case the first license is sold at a price $b^1 > \beta^1(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c^1 = \underline{c}$ to the winner of the license. Similarly, for $b^1 < \beta^1(\bar{c})$, we can assume that $c^1 = \bar{c}$.

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) &= \pi^{NS}(c_i, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&- \pi^{NS}(c_i, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&- \beta^2(c_i|\tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&+ \beta^2(\tilde{c}_i|\tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i).
\end{aligned}
$$

The difference between the last two terms equals to

$$
\begin{aligned}
&\beta^2(\tilde{c}_i|\tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i|\tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \quad = \\
&\qquad \int_{\tilde{c}_i}^{c_i} \pi^{NS}(c^2, \tilde{c}_i)\,(N-2)[1-F(c^2)]^{N-3} f(c^2)\, dc^2\ (N-1) f(\tilde{c}_i)
\end{aligned}
$$

and since $\pi^{NS}(c_i, c^1) < \pi^{NS}(c^2, c^1)$ for all $c^2 \in [\tilde{c}_i, c_i]$,

$$
\begin{aligned}
&\beta^2(\tilde{c}_i, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \quad > \\
&\qquad - \pi^{NS}(c_i, \tilde{c}_i)\ (N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&\qquad + \pi^{NS}(c_i, \tilde{c}_i)\ (N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i).
\end{aligned}
$$

Hence, we can conclude that

$$
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \ \geq\ 0,
$$

with equality only when $\tilde{c}_i = c_i$.

Similarly, if firm i mimics a marginal cost $\tilde{c}_i \geq c_i$, then, by changing its bid marginally, it will change its expected payoff by

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) &= -\ \beta^2(\tilde{c}_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&\quad - \frac{d}{d\tilde{c}_i}\{\,[1-F(\tilde{c}_i)]^{N-1}\,\beta^1(\tilde{c}_i)\,\}.
\end{aligned}
$$

After substituting the second term, we get

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\leq\; 0,$$

with equality only when $\tilde{c}_i = c_i$.

We have therefore shown that the derivative of the firm's expected profit is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as required for the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

To show that the bidding strategy $\beta^1(c_i)$ is strictly decreasing, notice that we can write its derivative as

$$\frac{d\beta^1}{dc_i}(c_i) \;=\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[\beta^2(c_i \,|\, c_i) + \int_{c_i}^{\bar{c}} \beta^2(c^2 \,|\, c^2) \frac{(N-1)[1-F(c^2)]^{N-2} f(c^2)}{[1-F(c_i)]^{N-1}} dc^2\right]$$

and, since Assumption 1 implies that $\beta^2(c_i \,|\, c_i)$ is decreasing, we can conclude that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[\,-\beta^2(c_i \,|\, c_i) + \beta^2(c_i \,|\, c_i)\,\right] \;=\; 0,$$

as required for the bidding strategy β^1 to be strictly decreasing.

□

Proof of Proposition 4:
First, notice that by rearranging the terms of the equation relating the bidding strategies $\beta(c_i)$, $\beta^1(c_i)$ and $\beta^2(c_i|c^1)$, given in the proof of Proposition 3, we get

$$[1-F(c_i)]^{N-1} \;\left[\,\beta^1(c_i) - \beta(c_i)\,\right] \;=\; (N-1)[1-F(c_i)]^{N-2} F(c_i) \times \left[\beta(c_i) - \int_{\underline{c}}^{c_i} \beta^2(c_i|c^1)\, \frac{f(c^1)}{F(c_i)}\, dc^1\right].$$

Therefore, for the entire result, it suffices to show that $\beta^1(c_i) > \beta(c_i)$.

Using the definitions of the strategies $\beta(c_i)$ and $\beta^1(c_i)$, we can show, by means of a direct calculation, that $\beta^1(c_i) > \beta(c_i)$ if and only if

$$\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi^{NS}(c^2, c^1) \; \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^2 \, dc^1 \quad >$$

$$\int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} \pi^{NS}(c^2, c^1) \; \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)} \, dc^2 \, dc^1.$$

That is, we shall show that

$$\mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, \pi^{NS}(c^2, c^1) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \;>\; \mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, \pi^{NS}(c^2, c^1) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,].$$

Since the market profit function $\pi^{NS}(c^2, c^1)$ is increasing in c^1, we have

$$\begin{aligned}
&\mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, \pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[\pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad > \; \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[\pi^{NS}(c^2_{-i}, c_i) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}}[\, \pi^{NS}(c^2_{-i}, c_i) \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[\pi^{NS}(c^2_{-i}, c_i) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad > \; \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[\pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\qquad = \; \mathbb{E}_{c^1_{-i}, c^2_{-i}}[\, \pi^{NS}(c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,],
\end{aligned}$$

as required for the result.

□

Proof of Proposition 5:
It is straightforward to verify that the function $\beta(c_i)$ is a solution to the differential equation that resulted from the necessary first-order condition. In addition, by using L'Hospital's rule, it is easy to check that

$$\lim_{c_i \to \bar{c}} \beta(c_i) = \int_{\underline{c}}^{\bar{c}} \pi(\bar{c} \,|\, \bar{c}, c) \, f(c) \, dc,$$

as required by the boundary condition.

Since the equation that produced the strategy $\beta(c_i)$ was only a necessary condition, we still need to establish that it is optimal for any bidder i with marginal cost c_i to bid $b_i = \beta(c_i)$, if all other bidders follow this bidding strategy.

Suppose that firm i bids $\tilde{b}_i = \beta(\tilde{c}_i)$, for $\tilde{c}_i \in [\underline{c}, \bar{c}]$ while having a marginal cost c_i.[26] Then, by changing its bid marginally, that is, by mimicking a marginally different type, it can change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; -\frac{d}{d\tilde{c}_i}\{\, \mathbb{P}[c^2_{-i} \geq \tilde{c}_i]\, \beta(\tilde{c}_i)\,\} \\
&- \int_{\underline{c}}^{\tilde{c}_i} \pi(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)(N-2)\,[1-F(\tilde{c}_i)]^{N-3}\, f(\tilde{c}_i) f(c^1)\ dc^1 \\
&+ \int_{\underline{c}}^{\tilde{c}_i} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)\,[1-F(\tilde{c}_i)]^{N-2}\, f(c^1)\ dc^1 \\
&+ \int_{\tilde{c}_i}^{\bar{c}} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\ (N-1)\,[1-F(c^1)]^{N-2}\, f(c^1)\ dc^1.
\end{aligned}$$

Substituting $\beta(c_i)$ and gathering the corresponding terms together result to

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} [\pi(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi(c_i \,|\, \tilde{c}_i, c^1)]\,(N-1)(N-2)[1-F(\tilde{c}_i)]^{N-3} f(\tilde{c}_i) f(c^1)\, dc^1 \\
&- \int_{\underline{c}}^{\tilde{c}_i} [\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi_2(c_i \,|\, \tilde{c}_i, c^1)]\ (N-1)\,[1-F(\tilde{c}_i)]^{N-2}\, f(c^1)\ dc^1 \\
&- \int_{\tilde{c}_i}^{\bar{c}} [\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi_2(c_i \,|\, \tilde{c}_i, c^1)]\ (N-1)\,[1-F(c^1)]^{N-2}\, f(c^1)\ dc^1.
\end{aligned}$$

Since the functions $\pi(c_i \,|\, \tilde{c}_i, c^1)$ and $-\pi_2(c_i \,|\, \tilde{c}_i, c^1)$ are decreasing in c_i, the change in the firm's expected profit is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

showing that the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ attains its maximum at $\tilde{c}_i = c_i$.

To show that the strategy $\beta(c_i)$ is decreasing, we can calculate its derivative to be

[26] If a license is sold at a price $b > \beta(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c = \underline{c}$ to the winner of that license. Similarly, for $b < \beta(\bar{c})$, we can assume that $c = \bar{c}$. In either case, no firm can profit from mimicking a type $\tilde{c}_i \notin [\underline{c}, \bar{c}]$.

$$\frac{d\beta}{dc_i}(c_i) \;=\; \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \;\times\; \left[-v(c_i) + \int_{c_i}^{\bar{c}} v(c^2) \frac{-\frac{d}{dc^2}\mathbb{P}[c_{-i}^2 \geq c^2]}{\mathbb{P}[c_{-i}^2 \geq c_i]}\, dc^2 \right],$$

where

$$\begin{aligned} v(c_i) \;&=\; \int_{\underline{c}}^{c_i} \pi(c_i \,|\, c_i, c^1) \times \frac{f(c^1)}{F(c^1)}\, dc^1 \\ &+\; \int_{\underline{c}}^{c_i} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1 - F(c_i)}{(N-2)f(c_i)} \right] \times \frac{f(c^1)}{F(c^1)}\, dc^1 \\ &+\; \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1 - F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2} F(c_i)}\, dc^1. \end{aligned}$$

Therefore, if the function $v(c_i)$ is decreasing, we can conclude that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{-\frac{d}{dc_i}\mathbb{P}[c_{-i}^2 \geq c_i]}{\mathbb{P}[c_{-i}^2 \geq c_i]} \times [\, -v(c_i) + v(c_i) \,] \;=\; 0$$

For the monotonicity of the function $v(c_i)$, it suffices to show that each of the three terms in its definition is decreasing. We demonstrate the result for the third term only, since the argument for the first two integrals is similar.

By rewriting this term as

$$\int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{[1 - F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^1,$$

we can calculate its derivative to be

$$\begin{aligned} &\frac{d}{dc_i} \left\{ \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1 - F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2} F(c_i)}\, dc^1 \right\} \;= \\ &\qquad \pi_2(c_i \,|\, c_i, c_i)\, \frac{1 - F(c_i)}{(N-2)f(c_i)} \\ &\qquad + \int_{c_i}^{\bar{c}} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{[1 - F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^1 \\ &\qquad + \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1)\, \frac{1 - F(c_i)}{(N-2)F(c_i)} \right] \times \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^1. \end{aligned}$$

By integrating the last term by parts, we get

$$\frac{d}{dc_i}\left\{ \int_{c_i}^{\bar c} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2}F(c_i)} \; dc^1 \right\} \;=$$

$$\int_{c_i}^{\bar c} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{[1-F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^1$$

$$+ \int_{c_i}^{\bar c} \frac{d}{dc^1} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-2)F(c_i)} \right] \times \frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}} \; dc^1,$$

and, since the assumption of the decreasing inverse hazard ratio $\frac{1-F(c_i)}{f(c_i)}$ implies that the term

$$\frac{[1-F(c_i)]^2}{(N-1)(N-2)F(c_i)f(c_i)} \;=\; \frac{[1-F(c_i)]}{(N-1)F(c_i)} \, \frac{[1-F(c_i)]}{(N-2)f(c_i)}$$

is also decreasing, we can drop a negative term from the first integral, so as to get

$$\frac{d}{dc_i}\left\{ \int_{c_i}^{\bar c} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2}F(c_i)} \; dc^1 \right\} \;\leq$$

$$\int_{c_i}^{\bar c} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \frac{[1-F(c_i)]^2}{(N-2)F(c_i)f(c_i)} \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \; dc^1$$

$$+ \int_{c_i}^{\bar c} \frac{d}{dc^1} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \frac{1-F(c_i)}{(N-2)F(c_i)} \times \frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}} \; dc^1.$$

Using again the assumption of the decreasing inverse hazard rate,

$$\frac{d}{dc_i}\left\{ \int_{c_i}^{\bar c} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-2)f(c_i)} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-2}F(c_i)} \; dc^1 \right\} \;\leq$$

$$\int_{c_i}^{\bar c} \frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \frac{1-F(c_i)}{(N-2)F(c_i)} \times \frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}} \; dc^1$$

$$+ \int_{c_i}^{\bar c} \frac{d}{dc^1} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \frac{1-F(c_i)}{(N-2)F(c_i)} \times \frac{[1-F(c^1)]^{N-1}}{[1-F(c_i)]^{N-1}} \; dc^1.$$

Finally, since

$$\frac{d}{dc_i} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \;+\; \frac{d}{dc^1} \left[-\pi_2(c_i \,|\, c_i, c^1) \right] \;=$$

$$-\left[\pi_{21}(c_i \,|\, c_i, c^1) + \pi_{22}(c_i \,|\, c_i, c^1) + \pi_{23}(c_i \,|\, c_i, c^1) \right] \;<\; 0,$$

for all marginal costs $c_i \in [\underline{c}, \bar{c}]$ and all $c^1 \in [c_i, \bar{c}]$, we can conclude that the derivative is negative, as desired.

The argument establishing that the stronger of the two oligopolists always makes a positive profit, unlike the weaker oligopolist who may regret his participation to the market, is identical to that in the proof of Corollary 1, with $\pi(c_i \,|\, c_i, c^1)$ in place of $\pi^{NS}(c_i, c^1)$. It is therefore omitted.

□

Proof of Lemma 3:
For arbitrary $c^1 \in [\underline{c}, \bar{c}]$, suppose that all firms follow the bidding strategy $\beta^2(\,.\,|\,c^1)$ and consider firm i with marginal cost $c_i \in [c^1, \bar{c}]$. Obviously, firm i cannot profit by bidding above $\beta^2(c^1 \,|\, c^1)$ or below $\beta^2(\bar{c} \,|\, c^1)$.[27] So, suppose that firm i mimics a type $\tilde{c}_i \in [c^1, \bar{c}]$, that is, it bids $\beta^2(\tilde{c}_i \,|\, c^1)$. Then its expected payoff will be

$$\Pi(\tilde{c}_i \,|\, c_i) = [1 - \tilde{F}(\tilde{c}_i)]^{N-2}\, [\, \pi(c_i \,|\, \tilde{c}_i, c^1) - \beta^2(\tilde{c}_i|c^1)\,].$$

By changing its bid marginally, firm i will change its expected payoff by

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;=\; & -\, \pi(c_i \,|\, \tilde{c}_i, c^1)\; (N-2)[1 - \tilde{F}(\tilde{c}_i)]^{N-3} \tilde{f}(\tilde{c}_i) \\ & +\, \pi_2(c_i \,|\, \tilde{c}_i, c^1)\; [1 - \tilde{F}(\tilde{c}_i)]^{N-2} \\ & -\, \frac{\partial}{\partial \tilde{c}_i} \{\, [1 - \tilde{F}(\tilde{c}_i)]^{N-2}\, \beta^2(\tilde{c}_i \,|\, c^1)\, \}. \end{aligned}$$

By calculating the derivative in the last term, we get

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;=\; & [\, \pi(\tilde{c}_i \,|\, \tilde{c}_i, c^1) - \pi(c_i \,|\, \tilde{c}_i, c^1)\,]\; (N-2)[1 - \tilde{F}(\tilde{c}_i)]^{N-3} \tilde{f}(\tilde{c}_i) \\ +\; & [\, \pi_2(c_i \,|\, \tilde{c}_i, c^1) - \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,]\; [1 - \tilde{F}(\tilde{c}_i)]^{N-2}. \end{aligned}$$

Since both the profit function $\pi(c_i \,|\, \tilde{c}_i, c^1)$ and the derivative $-\pi_2(c_i \,|\, \tilde{c}_i, c^1)$ are decreasing in c_i, it follows that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

[27] If the license is sold at a price $b^2 > \beta^2(c^1|c^1)$, an event outside the equilibrium path, we can assume that the remaining firms, in particular, the competing oligopolist, will attribute a marginal cost $c^2 = c^1$ to the winner of the license. Similarly, for $b^2 < \beta^2(\bar{c}|c^1)$, we can assume that $c^2 = \bar{c}$.

as required for the optimality of bidding $\beta^2(c_i \,|\, c^1)$.

In addition, when firm i has marginal cost $c_i \in [\underline{c}, c^1]$, the previous analysis shows that $\Pi_1(\tilde{c}_i \,|\, c_i) < 0$, for all $\tilde{c}_i \in [c^1, \bar{c}]$. Hence, firm i is best-off bidding $\beta^2(c^1 \,|\, c^1)$.

Finally, for the monotonicity of the strategy $\beta^2(c_i \,|\, c^1)$, notice that, since the inverse hazard rate $[1 - F(c)]/f(c)$ is decreasing, the expression

$$v(c \,|\, c, c^1) \;=\; \pi(c \,|\, c, c^1) \;-\; \pi_2(c \,|\, c, c^1)\, \frac{1 - F(c)}{(N-2) f(c)}$$

is decreasing in c. Therefore, the derivative

$$\begin{aligned} \frac{\partial \beta^2}{\partial c_i}(c_i \,|c^1) \;&=\; -\, v(c_i \,|\, c_i, c^1)\, \frac{(N-2) f(c_i)}{1 - F(c_i)} \\ &+ \int_{c_i}^{\bar{c}} v(c \,|\, c, c^1)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \,\times\, \frac{(N-2) f(c_i)}{1 - F(c_i)} \end{aligned}$$

is negative, showing that the strategy $\beta^2(c^1 \,|\, c^1)$ is strictly decreasing in $c_i \in (c^1, \bar{c}\,]$.

□

Proof of Lemma 4:
The derivative of $\beta^2(c_i \,|\, c_i)$ equals to

$$\begin{aligned} \frac{d}{dc_i}[\, \beta^2(c_i \,|\, c_i)\,] \;&=\; -v(c_i, c_i)\, \frac{(N-2) f(c_i)}{1 - F(c_i)}, \\ &+ \int_{c_i}^{\bar{c}} v_2(c, c_i)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \\ &+ \int_{c_i}^{\bar{c}} v(c, c_i)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \,\times\, \frac{(N-2) f(c_i)}{1 - F(c_i)}, \end{aligned}$$

or, after integrating the last term by parts, to

$$\begin{aligned} \frac{d}{dc_i}[\, \beta^2(c_i \,|\, c_i)\,] \;&=\; \int_{c_i}^{\bar{c}} v_2(c, c_i)\, \frac{(N-2)[1 - F(c)]^{N-3} f(c)}{[1 - F(c_i)]^{N-2}}\, dc \\ &+ \int_{c_i}^{\bar{c}} v_1(c, c_i)\, \frac{[1 - F(c)]^{N-2}}{[1 - F(c_i)]^{N-2}}\, dc \,\times\, \frac{(N-2) f(c_i)}{1 - F(c_i)}. \end{aligned}$$

For all $c_i \in [\underline{c}, \bar{c}]$, since the inverse hazard ratio $[1 - F(c)]/f(c)$ is decreasing, we have

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i\,|\,c_i)\,] \;&<\; \int_{c_i}^{\bar c} v_2(c\,|\,c_i)\,\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+\; \int_{c_i}^{\bar c} \tilde v_1(c\,|\,c_i)\,\frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\,dc\,\frac{(N-2)f(c_i)}{1-F(c_i)},
\end{aligned}
$$

where

$$
\tilde v_1(c_i\,|\,c^1) = \frac{d\pi}{dc_i}(c_i\,|\,c_i,c^1) - \frac{d\pi_2}{dc_i}(c_i\,|\,c_i,c^1)\,\frac{1-F(c_i)}{(N-2)f(c_i)}.
$$

Since the expression

$$
\begin{aligned}
w(c,c_i) \;&=\; \frac{v_2(c\,|\,c_i)}{-\tilde v_1(c\,|\,c_i)} - \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]} \\
&=\; \frac{1}{2} - \frac{f(c_i)/[1-F(c_i)]}{f(c)/[1-F(c)]}
\end{aligned}
$$

is negative for $c = c_i$, positive for $c = \bar c$, continuous and increasing with respect to $c \in [c_i, \bar c]$, there exists a value $c^* = c^*(c_i) \in (c_i, \bar c)$ such that

$$
w(c,c_i)\;\begin{cases} <0, & \text{for } c \in [c_i, c^*); \\ =0, & \text{for } c = c^*; \\ >0, & \text{for } c \in (c^*, \bar c\,]. \end{cases}
$$

Therefore, since the function $-\tilde v_1(c, c^1)$ is decreasing in $c \in [c_i, \bar c]$ and positive, we have

$$
\begin{aligned}
\frac{d}{dc_i}[\,\beta^2(c_i\,|\,c_i)\,] \;&<\; \int_{c_i}^{c^*} -\tilde v_1(c\,|\,c_i)\; w(c,c_i)\;\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&+\; \int_{c^*}^{\bar c} -\tilde v_1(c\,|\,c_i)\; w(c,c_i)\;\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \\
&<\; -\tilde v_1(c^*\,|\,c_i)\;\int_{c_i}^{\bar c} w(c,c_i)\;\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc
\end{aligned}
$$

Assumption 2 implies that

$$
\int_{c_i}^{\bar c} w(c,c_i)\;\frac{(N-2)[1-F(c)]^{N-3}f(c)}{[1-F(c_i)]^{N-2}}\,dc \;\le\; 0,
$$

which suffices for $\beta^2(c_i|c_i)$ to be decreasing.

□

Proof of Proposition 6:
Suppose that all firms follow the bidding strategy (β^1, β^2) and consider firm i with marginal cost $c_i \in [\underline{c}, \bar{c}]$. The optimality of bidding $\beta^2(c_i \,|\, c^1)$ in the second auction, following the sale of the first license at a price b^1 corresponding to a marginal cost $c^1 = (\beta^1)^{-1}(b^1)$, has been established in Lemma 3.[28] Therefore, we only need to examine the optimality of bidding $\beta^1(c_i)$ in the first auction.

Obviously, firm i cannot gain from submitting a bid above $\beta^1(\underline{c})$ or below $\beta^1(\bar{c})$. So, suppose that it mimics a type $\tilde{c}_i \in [\underline{c}, \bar{c}]$, that is, it bids $\beta^1(\tilde{c}_i)$. If $\tilde{c}_i \leq c_i$, then, by changing its bid marginally, firm i will change its expected payoff by

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi(c_i \,|\, c_i, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1, \\
&-\; \beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \frac{d}{d\tilde{c}_i}\{\, [1-F(\tilde{c}_i)]^{N-1}\, \beta^1(\tilde{c}_i)\,\}.
\end{aligned}$$

After substituting the appropriate expression for the last term, the change in the expected payoff of firm i becomes

$$\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;&=\; \pi(c_i \,|\, c_i, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
&+\; \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
&-\; \beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&+\; \beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i).
\end{aligned}$$

The difference between the last two terms equals to

[28] In case the first license is sold at a price $b^1 > \beta^1(\underline{c})$, an event outside the equilibrium path, we can assume that the remaining firms attribute a marginal cost $c^1 = \underline{c}$ to the winner of the license. Similarly, for $b^1 < \beta^1(\bar{c})$, we can assume that $c^1 = \bar{c}$.

$$
\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\,(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \;=\\
&\qquad \int_{\tilde{c}_i}^{c_i} \pi(c^2 \,|\, c^2, \tilde{c}_i)\,(N-2)[1-F(c^2)]^{N-3} f(c^2)\, dc^2 \;(N-1) f(\tilde{c}_i) \\
&\qquad + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i).
\end{aligned}
$$

Since $\pi(c_i \,|\, \tilde{c}_i, c^1)$ is decreasing in c_i, we have

$$
\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \;>\\
&\qquad \int_{\tilde{c}_i}^{c_i} \pi(c_i \,|\, c^2, \tilde{c}_i)\;(N-2)[1-F(c^2)]^{N-3} f(c^2)\, dc^2 \;(N-1) f(\tilde{c}_i) \\
&\qquad + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i)
\end{aligned}
$$

and, by integrating the first term by parts,

$$
\begin{aligned}
&\beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&-\beta^2(c_i \,|\, \tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \;>\\
&\qquad \pi(c_i \,|\, \tilde{c}_i, \tilde{c}_i)\;(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
&\qquad - \;\pi(c_i \,|\, c_i, \tilde{c}_i)\;(N-1)[1-F(c_i)]^{N-2} f(\tilde{c}_i) \\
&\qquad - \int_{\tilde{c}_i}^{c_i} -\pi_2(c_i \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i) \\
&\qquad + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; \geq \;\; & - \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
& + \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
& - \int_{\tilde{c}_i}^{c_i} -\pi_2(c_i \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i) \\
& + \int_{\tilde{c}_i}^{c_i} -\pi_2(c^2 \,|\, c^2, \tilde{c}_i)\,[1-F(c^2)]^{N-2}\, dc^2 \;(N-1) f(\tilde{c}_i)
\end{aligned}
$$

and, since the derivative $-\pi_2(c_i \,|\, \tilde{c}_i, , c^1)$ is decreasing in c_i, we can conclude that

$$
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; \geq \;\; 0,
$$

with equality only when $\tilde{c}_i = c_i$.

Similarly, if firm i mimics a marginal cost $\tilde{c}_i \geq c_i$, then, by changing its bid marginally, it will change its expected payoff by

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; = \;\; & - \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1, \\
& - \beta^2(\tilde{c}_i \,|\, \tilde{c}_i)\,(N-1)[1-F(\tilde{c}_i)]^{N-2} f(\tilde{c}_i) \\
& - \frac{d}{d\tilde{c}_i}\{\, [1-F(\tilde{c}_i)]^{N-1}\, \beta^1(\tilde{c}_i)\,\}.
\end{aligned}
$$

By substituting the appropriate expression for the last term, we get

$$
\begin{aligned}
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; = \;\; & - \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1 \\
& + \int_{\tilde{c}_i}^{\bar{c}} -\pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\, dc^1,
\end{aligned}
$$

which implies that

$$
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;\; \leq \;\; 0,
$$

with equality only when $\tilde{c}_i = c_i$.

We have therefore shown that the derivative of the firm's expected profit is

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as required for the firm's expected profit $\Pi(\tilde{c}_i \,|\, c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

Finally, to show that the bidding strategy $\beta^1(c_i)$ is strictly decreasing, notice that we can write its derivative as

$$\frac{d\beta^1}{dc_i}(c_i) \;=\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times \left[-v^1(c_i) + \int_{c_i}^{\bar{c}} v^1(c^2) \, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^2)}{[1-F(c_i)]^{N-1}} \, dc^2 \right],$$

where

$$\begin{aligned} v^1(c_i) \;&=\; \beta^2(c_i \,|\, c_i) \\ &+\; \int_{c_i}^{\bar{c}} \left[-\pi_2(c_i \,|\, c_i, c^1) \, \frac{1-F(c_i)}{(N-1)f(c_i)} \right] \times \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^1. \end{aligned}$$

Because of Assumption 2, the term $\beta^2(c_i \,|\, c_i)$ is decreasing in c_i. In addition, by an argument similar to the one used for the corresponding term in $\frac{d}{dc_i}\beta(c_i)$, we can show that the second term is also decreasing in c_i. Therefore, the function v^1 is decreasing.

It follows that

$$\frac{d\beta}{dc_i}(c_i) \;<\; \frac{(N-1)f(c_i)}{1-F(c_i)} \times [\,-v(c_i) + v(c_i)\,] \;=\; 0,$$

as required for the bidding strategy β^1 to be strictly decreasing.

□

Proof of Proposition 8:
First, notice that by rearranging the terms of the equation relating the bidding strategies $\beta(c_i)$, $\beta^1(c_i)$ and $\beta^2(c_i \,|\, c^1)$, given in the proof of Proposition 3, we get

$$\begin{aligned} &[1-F(c_i)]^{N-1} \;\; [\,\beta^1(c_i) - \beta(c_i)\,] \;= \\ &\qquad (N-1)[1-F(c_i)]^{N-2} F(c_i) \;\left[\beta(c_i) - \int_{\underline{c}}^{c_i} \beta^2(c_i \,|\, c^1) \, \frac{f(c^1)}{F(c_i)} \, dc^1 \right]. \end{aligned}$$

Therefore, for the entire result, it suffices to show that $\beta^1(c_i) > \beta(c_i)$.

Using the definitions of the strategies $\beta(c_i)$ and $\beta^1(c_i)$, we can show, by means of a direct calculation, that $\beta^1(c_i) > \beta(c_i)$ if and only if

$$\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} v(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \quad +$$

$$\int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2 \,|\, c^2, c^1) \, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \quad >$$

$$\int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} v(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)} \, dc^2 \, dc^1,$$

where

$$v(c^2 \,|\, c^2, c^1) = \pi(c^2 \,|\, c^2, c^1) - \pi_2(c^2 \,|\, c^2, c^1) \frac{1-F(c^2)}{(n-2) f(c^2)}.$$

Since the second double integral is positive, it suffices to show that

$$\int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} v(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}} \, dc^2 \, dc^1 \quad >$$

$$\int_{\underline{c}}^{c_i} \int_{c_i}^{\bar{c}} v(c^2 \,|\, c^2, c^1) \, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{(N-1)[1-F(c_i)]^{N-2} F(c_i)} \, dc^2 \, dc^1,$$

that is, to show that

$$\mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \;>\; \mathbb{E}_{c^1_{-i}, c^2_{-i}} [\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,].$$

Since the function $v(c^2 \,|\, c^2, c^1)$ is increasing in c^1, we have

$$
\begin{aligned}
&\mathbb{E}_{c^1_{-i},c^2_{-i}}[\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c^1_{-i} \geq c_i \,] \\
&\quad = \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\quad > \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[v(c^2_{-i} \,|\, c^2_{-i}, c_i) \,|\, c^1_{-i} \in [c_i, c^2_{-i}]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\quad = \mathbb{E}_{c^2_{-i}}[\, v(c^2_{-i} \,|\, c^2_{-i}, c_i) \,|\, c^2_{-i} \geq c_i \,] \\
&\quad = \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[v(c^2_{-i} \,|\, c^2_{-i}, c_i) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\quad > \mathbb{E}_{c^2_{-i}}[\, \mathbb{E}_{c^1_{-i}}[v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^1_{-i} \in [\underline{c}, c_i]\,] \,|\, c^2_{-i} \geq c_i \,] \\
&\quad = \mathbb{E}_{c^1_{-i},c^2_{-i}}[\, v(c^2_{-i} \,|\, c^2_{-i}, c^1_{-i}) \,|\, c^2_{-i} \geq c_i \geq c^1_{-i} \,],
\end{aligned}
$$

as required for the result.

□

Proof of Proposition 9:
Similarly to the proof of Proposition 5, notice that the strategy $\beta(c_i)$ can be expressed as

$$
\beta(c_i) = \int_{c_i}^{\bar{c}} u(c^2 \,|\, c^2) \, \frac{(N-1)(N-2)\,[1-F(c^2)]^{N-3}\, F(c^2)\, f(c^2)}{\mathbb{P}[c^2_{-i} \geq c_i]} \, dc^2,
$$

where

$$
\begin{aligned}
u(\tilde{c} \,|\, c) &= \int_{\underline{c}}^{\tilde{c}} \pi(c \,|\, \tilde{c}, c^1) \times \frac{f(c^1)}{F(\tilde{c})} \, dc^1 \\
&\quad - \int_{\underline{c}}^{\tilde{c}} \left[\pi_2(c \,|\, \tilde{c}, c^1) \, \frac{1-F(\tilde{c})}{(N-2) f(\tilde{c})} \right] \times \frac{f(c^1)}{F(\tilde{c})} \, dc^1 \\
&\quad - \int_{c}^{\tilde{c}} \left[\pi_2(c \,|\, \tilde{c}, c^1) \, \frac{1-F(\tilde{c})}{(N-2) f(\tilde{c})} \right] \times \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(\tilde{c})]^{N-2} F(\tilde{c})} \, dc^1,
\end{aligned}
$$

for $\tilde{c}$, $c \in [\underline{c}, \bar{c}]$. In particular, $u(c) = u(c \,|\, c)$ is the valuation of a firm with marginal cost c, assuming that its market opponent is stronger.

Therefore, we have

$$
\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) = (N-1)(N-2)\,[1-F(\tilde{c}_i)]^{N-3}\, F(\tilde{c}_i)\, f(\tilde{c}_i)\; [\, u(\tilde{c}_i \,|\, \tilde{c}_i) - u(\tilde{c}_i \,|\, c_i) \,]
$$

and

$$\frac{\partial^2 \Pi}{\partial \tilde{c}_i \partial c_i}(\tilde{c}_i \,|\, c_i) \;=\; -\,(N-1)(N-2)\,[1-F(\tilde{c}_i)]^{N-3}\,F(\tilde{c}_i)\,f(\tilde{c}_i)\;u_2(\tilde{c}_i|c_i).$$

Hence, if we can show that

$$u_2(\tilde{c}_i \,|\, c_i) \;>\; 0,$$

then, since $\frac{\partial \Pi}{\partial \tilde{c}_i}(c_i \,|\, c_i) = 0$, we can conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \begin{cases} > 0, & \text{for } \tilde{c}_i < c_i; \\ = 0, & \text{for } \tilde{c}_i = c_i; \\ < 0, & \text{for } \tilde{c}_i > c_i, \end{cases}$$

as it suffices for firm i's expected profit function $\Pi(\tilde{c}_i|c_i)$ to attain its maximum at $\tilde{c}_i = c_i$.

Notice that

$$\begin{aligned} u_2(\tilde{c}_i \,|\, c_i) \;&=\; \int_{\underline{c}}^{\tilde{c}_i} \pi_1(c_i \,|\, \tilde{c}_i, c^1) \,\times\, \frac{f(c^1)}{F(\tilde{c}_i)}\, dc^1 \\ &-\; \int_{\underline{c}}^{\tilde{c}_i} \left[\pi_{21}(c_i \,|\, \tilde{c}_i, c^1)\, \frac{1-F(\tilde{c}_i)}{(N-2)f(\tilde{c}_i)} \right] \times\, \frac{f(c^1)}{F(\tilde{c}_i)}\, dc^1 \\ &-\; \int_{c_i}^{\bar{c}} \left[\pi_{21}(c_i \,|\, \tilde{c}_i, c^1)\, \frac{1-F(\tilde{c}_i)}{(N-2)f(\tilde{c}_i)} \right] \times\, \frac{[1-F(c^1)]^{N-2} f(c^1)}{[1-F(\tilde{c}_i)]^{N-2} F(\tilde{c}_i)}\, dc^1. \end{aligned}$$

Therefore, if N is sufficiently large, then the positive term dominates the negative ones, so that $u_2(\tilde{c}_i \,|\, c_i) > 0$. Moreover, it is possible to find N^* such that for $N > N^*$ we have $u_2(\tilde{c}_i \,|\, c_i) > 0$, for all $\tilde{c}_i, c_i \in [\underline{c}, \bar{c}]$, uniformly.

The rest of the proof is identical to that of Proposition 5, so, it is omitted.

□

Proof of Proposition 10:
Suppose, contrary to our assertion, that there exists a symmetric equilibrium in monotone bidding strategies (β^1, β^2). Since β^1 is strictly decreasing, the announcement of the first-round winning bid reveals the marginal cost c^1 of the strongest oligopolist. Therefore, in the second round, the firms update their beliefs, so that

$$c_i \sim \tilde{F}(c) = \frac{F(c) - F(c^1)}{1 - F(c^1)},$$

for all $c \in [c^1, \bar{c}]$.

If the number of firms participating in the second auction, $N-1$, is sufficiently large, so as to satisfy the inequality

$$\frac{\pi_2(c,c)}{\pi_{12}(c,c)} > \frac{[1-F(c)]}{(N-2)\, f(c)}\,,$$

for all $c \in [\underline{c}, \bar{c}]$, then, as shown in Das Varma [9], the strategy

$$\begin{aligned} \beta^2(c_i\,|c^1) &= \int_{c_i}^{\bar{c}} \pi(c^2\,|\,c^2, c^1)\, \frac{(N-2)[1-F(c^2)]^{N-3} f(c^2)}{[1-F(c_i)]^{N-2}}\, dc^2 \\ &+ \int_{c_i}^{\bar{c}} -\pi_2(c^2\,|\,c^2, c^1)\, \frac{[1-F(c^2)]^{N-2}}{[1-F(c_i)]^{N-2}}\, dc^2, \end{aligned}$$

for $c_i \geq c^1$, forms the unique equilibrium in the auction of the second license. In particular, for a marginal cost $c_i < c^1$, firm i bids $b^2 = \beta^2(c^1\,|c^1)$.

By assuming, as we did in the case of the Cournot oligopoly, that

$$\int_{c_i}^{\bar{c}} \frac{[1-F(c)]^{N-2}}{[1-F(c_i)]^{N-2}}\, dc \;\geq\; \sup_{c \geq c_i} \left\{ \frac{v_2(c, c_i)}{-\tilde{v}_1(c, c_i)} \right\} \frac{1-F(c_i)}{(N-2)\, f(c_i)},$$

for all $c_i \in [\underline{c}, \bar{c}]$, where

$$\tilde{v}_1(c, c^1) = \frac{d}{dc}[\pi(c\,|\,c, c^1)] - \frac{d}{dc}[\pi_2(c\,|\,c, c^1)]\, \frac{1-F(c)}{(N-2)\, f(c)}\,,$$

we can ensure that the strategy $\beta^2(c_i|c_i)$ is decreasing in the marginal cost c_i.

In the first auction, the optimization of the profit function $\Pi(\tilde{c}_i|c_i)$ of a firm with marginal cost c_i results in the strategy

$$\begin{aligned} \beta^1(c_i) &= \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} \pi(c^2\,|\,c^2, c^1)\, \frac{(N-1)(N-2)[1-F(c^2)]^{N-3} f(c^2) f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1 \\ &+ \int_{c_i}^{\bar{c}} \int_{c^1}^{\bar{c}} -\pi_2(c^2\,|\,c^2, c^1)\, \frac{(N-1)[1-F(c^2)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1 \\ &+ \int_{c_i}^{\bar{c}} \int_{c_i}^{c^1} -\pi_2(c^2\,|\,c^2, c^1)\, \frac{(N-1)[1-F(c^1)]^{N-2} f(c^1)}{[1-F(c_i)]^{N-1}}\, dc^2\, dc^1\,, \end{aligned}$$

as the unique solution of the differential equation derived by the necessary first-order condition $\Pi_1(c_i \,|\, c_i) = 0$.

To check sufficiency, we can calculate, for $\tilde{c}_i \geq c_i$,

$$\begin{aligned} \frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) &= \int_{\tilde{c}_i}^{\bar{c}} \pi_2(c_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1 \\ &- \int_{\tilde{c}_i}^{\bar{c}} \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)\,(N-1)[1-F(c^1)]^{N-2} f(c^1)\,dc^1. \end{aligned}$$

Since $\pi_2(c_i \,|\, \tilde{c}_i, c^1) > \pi_2(\tilde{c}_i \,|\, \tilde{c}_i, c^1)$, for $\tilde{c}_i \geq c_i$, we conclude that

$$\frac{\partial \Pi}{\partial \tilde{c}_i}(\tilde{c}_i \,|\, c_i) \;>\; 0,$$

showing that the firm's deviation from $\beta^1(c_i)$ to $\beta^1(\tilde{c}_i)$, for $\tilde{c}_i > c_i$, is profitable. Hence, the strategy $\beta^1(c_i)$ fails to support an equilibrium.

Since the strategy β^1 was the unique solution to the necessary condition, we conclude that the sequential auction has no equilibrium in strictly monotone strategies.

□

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