The Condorcet-Duverger trade-off: comparing plurality, runoff and approval elections.

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1 Introduction

Each electoral system has its own strengths and weaknesses. A major weakness common to all of them is that each electoral system features inefficient equilibria, in which voters are drawn to elect inferior candidates.

To show this, game-theoretic analyses start with the axiom, shared with economic analysis, that voters have a clear preference ranking among possible outcomes. These preferences guide their actions in the voting game. This axiom is a natural starting point to check how individual preferences are aggregated through elections. However, a widespread feature of real-life elections is that many voters are somehow uncertain about which candidate they "should" prefer: who best fits their needs?

Encompassing preference uncertainty in the analysis requires a partial departure from the axiom that voters have a clear ranking among outcomes. We propose a model based on Myerson's (1998) *extended Poisson games* to compare the properties of three electoral systems when the voters' preferred outcome depends on a "state of nature" that is uncertain.

In an election with three candidates, we show that *plurality* and *run-off* elections feature dominated equilibria. They are dominated because they elected the candidate preferred by a majority of the population only with a low probability. By contrast, *approval elections* feature a unique equilibrium, in which the best candidate is elected with a probability that approaches 1 as population size increases. The contrast between the systems results from what we call the *Condorcet-Duverger* trade-off.

Duverger's Law predicts that only two candidates should receive votes in plurality elections. The game-theoretic version of that law works as follows: assume there are three candidates: A,

B and C, and each voter has a strict ranking across candidates. The Duvergerian prediction is that no voter will cast his ballot on the candidate who is expected to rank third in the election – an expectation that is therefore self-fulfilling. The reason is that the voter's single ballot has virtually no chance of being pivotal in favor of the third candidate, even in comparison to the pivot probabilities among the top-two candidates.

In contrast, the *Extended Condorcet Jury Theorem* shows that –in two-candidate elections at least– another type of equilibrium can arise. Such *Condorcet-like* equilibria ensure that the best candidate is elected at all times. As shown for instance by Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997) and Myerson (1998), voters with uncertain preferences may indeed prefer to adopt a strategy of splitting their votes among the two candidates, to make sure that the best one is elected. Yet, these results do not directly extend to three-candidate elections, because of the Duvergerian forces that induce voters to concentrate their votes on the top-two candidates, even when they prefer the third one.

A first contribution of our analysis is to show how the forces that shape *Condorcet-like* and *Duvergerian* equilibria interact with one another in such a three-candidate setting. We show that Duvergerian equilibria exist both in plurality and runoff elections, despite the fact that runoff elections are sometimes viewed as a solution to the problem that "third candidates" cannot compete against dominant parties. In our model this result means that only candidate A or only candidate B can win the election, even though A is should only win in the state of nature a, whereas B should win in state b. Condorcet-like equilibria, when they exist, dominate such Duvergerian equilibria.

Our second, and more important, contribution is to show that these two electoral systems are therefore dominated by a third system: *approval voting*, in which voters can cast a ballot on as many candidates as they like. The dominance of approval voting results from the fact that it has no Duvergerian equilibrium, but a unique, Condorcet-like, equilibrium. Like in the Extended Condorcet Jury Theorem, in equilibrium, voters necessarily split their votes across parties in such a way that A wins in state a, B wins in state b, whereas the dominated candidate C always ranks third in the election. Therefore, under approval voting, the forces that shape Duvergerian equilibria are swamped by those that allow efficient learning by the electorate.

2 The model

We compare three electoral systems: Plurality, Run-off and Approval Voting. In plurality and approval elections, the candidate receiving the most votes is elected. In case of a tie, we assume that the candidate ranked alphabetically first is elected (results would not change if ties were broken by a coin toss). In the run-off electoral system analysed here, a candidate is elected in the first round if he receives more than 50% of the votes. If no candidate receives 50% in the first round, then a second round is organized among the two candidates receiving the most votes. Throughout the paper, we call such two candidates the “top-two candidates”.

We conduct our analysis under the assumption that the number of voters in the electorate is distributed according to a Poisson distribution (see, e.g. Myerson 2000 for the properties of Poisson Games). There are two states of nature: a and b. Conditional on the state of nature, voters’ types are attributed by iid draws. We are therefore analysing an *extended Poisson Game* as introduced by Myerson (1998).

2.1 The voters

There are three candidates, $P = A$, B and C, three types of voters, $t \in \{t_A, t_B, t_C\}$ and two states of nature: $\omega \in \{a, b\}$

- $T = \{t_A, t_B, t_C\}$: set of types
- $r\left(t|\omega\right)$: probability that a player is attributed type t if state of the world is ω. $\sum_t r\left(t|\omega\right) = 1$, $\forall \omega$.
- We denote the utility of the voters by the function $U\left(P, t, \omega\right)$, where P is the party winning the election, t is the type, and ω is the state of nature. Voters do not directly derive a benefit from the ballot they cast.

A minority of the voters are “partisans”. These are types t_C. The remainder of the population, i.e. types t_A and t_B, have state-contingent preferences.

- $r\left(t_C|a\right) = r\left(t_C|b\right) < 1/2$. Types t_C are called the *minority block*: in expected terms, they represent a minority of the electorate. They are partisans in the sense that they prefer candidate C, independently of the state of nature. For expositional simplicity, we also assume that they are indifferent between the other two candidates:

$$\begin{aligned} U\left(P, t_C, \omega\right) &= 1 \text{ if } P = C \\ &= 0 \text{ if } P \in \{A, B\}. \end{aligned}$$

- Together, types t_A and t_B represent a majority of the population. Ex ante, they have different priors about which candidate, A or B, best represents their interest. However, conditional on the state of nature, their preferences are aligned: they prefer A to B in state a and they prefer B to A in state b. In both states of nature, they perceive candidate C as being the worst option: for types $t \in \{t_A, t_B\}$, we have:

$$\begin{aligned} U(P,t,\omega) &= 1 \text{ if } (P,\omega) = (A,a) \text{ or } (B,b) \\ &= 0 \text{ if } (P,\omega) = (A,b) \text{ or } (B,a) \\ &= -1 \text{ if } P = C \end{aligned}$$

- Each state of nature occurs with probability 1/2. The difference between types t_A and t_B stems from the fact that $r(t_A|a) > r(t_A|b)$. That is, there are more voters with type t_A in state a than in state b. The only information available to voters is their type. Knowing their type, they form beliefs about the probability of each state of nature. By Bayesian updating, a type t voter infers that:
$$q(\omega|t) = \frac{r(t|\omega)}{r(t|a) + r(t|b)}.$$
Since $r(t_A|a) > r(t_B|b)$, we have:
$$q(a|t_A) > \frac{1}{2} > q(b|t_A) \text{ and } q(b|t_B) > \frac{1}{2} > q(a|t_B).$$

- Finally, we make the technical assumption that, on average, there are more types t_A than types t_B in the population:
$$r(t_A|a) + r(t_A|b) \geq r(t_B|a) + r(t_B|b).$$
The signals are therefore biased. This assumption is made to make sure that our results cannot depend on any type of symmetry across the distribution of types.

- `We show in section [TBW] that our results remain valid even if type-C voters were not indifferent between` A `and` B.

2.2 Further notation

- n : Expected number of players in the game. The actual number of players will be a draw from a Poisson distribution with parameter n

- The action set of the voters is denoted by $\Psi = \{A, B, C\}$ in plurality and in run-off elections.[1] That is, a voter must cast his ballot on one of the three candidates. Clearly, equilibrium strategies can be strictly mixed. Note that Ψ does not include the abstention ballot. Our results do not hinge on that assumption.

- $x = (x(\psi))_{\psi \in \Psi} \in Z(\Psi) \subset \mathcal{N}_+^{\Psi}$: *action profile*, that is the number of players who choose action ψ. $Z(\Psi)$ is the set of all possible action profiles

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[1] Under approval voting, the action set is enlarged. We return to this when we turn to approval elections.

- $\sigma(\psi|t)$: (mixed-)*strategy function* for type t. $\sum_{\psi} \sigma(\psi|t) = 1$, $\forall t$
- $\tau(\psi|\omega) = \sum_t r(t|\omega)\ \sigma(\psi|t)$ is the conditional probability that a randomly sampled voter chooses action ψ. $\tau(\omega) = (\tau(\psi|\omega))_{\psi\in\Psi}$ is the *expected fraction* of players choosing each action ψ in state ω
- $n\ \tau(\psi|\omega)$, $\forall\psi \in \Psi$, $\forall\omega \in \Omega$: *expected results, given the state.* This is the expected *number* of players choosing action ψ in the state of nature ω, if the strategy function is $\sigma(\psi|t)$

3 Plurality Elections

The purpose of this section is to introduce the main forces and trade-offs that shape electoral outcomes. Plurality elections have been widely analyzed, and this section only reproduces results that have already been proven in the literature. We begin by reminding the reader that, under plurality elections, Duvergerian equilibria always exist. In our setup, the meaning of this result is that, if either candidate A or B is expected to rank third in the election, he will receive no vote. Therefore, his probability of winning the election is zero, in both states of nature. This is an inferior equilibrium, since a majority of the population would like A (respectively B) to be elected in state a (resp. b). Even if widely known, it is important to begin with this result to introduce the payoff functions of the voters, and highlight two central properties, introduced by Myerson (2000), that shape them.

Let $\lambda(P)$ denote the *realized* vote shares of each candidate $P \in \{A, B, C\}$. Under plurality voting, the winner of the election is the candidate with the highest vote share, and voters can cast their ballot on exactly one candidate (we do not allow for abstention. `See section [TBW] for a discussion of that restriction`).

3.1 Duvergerian equilibria always exist

The problem of the voters is to choose for which party to vote. Voters are instrumentally motivated: they vote for a party if that vote maximizes their expected utility. Under plurality elections, a candidate wins if he has more votes than the other two candidates. That is, candidate, say, A wins if $x(A) \geq \max[x(B), x(C)]$. Of course, your single vote is only valuable if it is pivotal. That is if your own vote changes the outcome of the election. For instance, if A trails behind B by exactly 1 vote (and is ahead of C): $x(A) = x(B) - 1 \geq x(C)$, then a single vote for A makes the difference: with your vote, A is elected instead of B. The event that a vote for candidate P is pivotal against another candidate Q is denoted piv_{PQ}.

Of course, the value of a ballot for either candidate depends on the *probability* that it is pivotal. This *pivot probability* depends on the fraction of (other) voters in the population who vote for each candidate. The problem of a voter is thus to identify which ballot has the highest expected value, given the expected action profile of the other voters. An equilibrium is reached when such an expected action profile is sustained by each individual voter's best response. That is, an equilibrium is reached if, for a given $\tau(\omega)$, the fraction of voters who choose action ψ is indeed $\tau(\psi|\omega)$ in state ω.

Proposition 1 *(Riker 1982, Myerson and Weber 1993) Duvergerian equilibria always exist under plurality voting. That is,* $\lambda(A) = 0$ *or* $\lambda(B) = 0$, $\forall\omega$ *are (self-fulfilling) equilibria under plurality elections. In our model, these equilibria imply that the majority block faces a probability 1/2 of electing their best candidate.*

The rationale behind this result is well known: under plurality voting, instrumentally motivated voters avoid wasting their ballot. Therefore, they concentrate their votes on the candidates who are perceived as *serious contenders* in the electoral race. This implies that any prior belief that either A or B is "weak" induces all voters in the majority block to abandon that candidate.

Although this result is standard, it is interesting to highlight the forces behind it, since they lie at the heart of all our subsequent results. The first step is to identify the value of a ballot. As is known since Riker and Ordeshook (1968), it depends on the probability that the ballot is pivotal between two candidates. The second step is to pinpoint pivot probabilities and to show how they imply that a vote for a trailing candidate would be a wasted ballot.

Before starting this, one must remember that the expected utility of a majority-block voter $t \in \{t_A, t_B\}$ is:

$$\begin{aligned} EU(t) = \quad & q(a|t) \quad \{\Pr(\lambda(A) \geq \max[\lambda(B), \lambda(C)]) - \Pr(\lambda(C) > \max[\lambda(A), \lambda(B)])\} \\ & +q(b|t) \quad \{\Pr(\lambda(B) \geq \max[\lambda(A), \lambda(C)]) - \Pr(\lambda(C) > \max[\lambda(A), \lambda(B)])\}. \end{aligned} \tag{1}$$

This reads as follows. A type t expects that the state of nature is a with probability $q(a|t)$. In that case, the voter's utility is 1 if A wins, and -1 if C wins (it takes value 0 if B wins). This is the first line of (1). Conversely, the state of nature is b with probability $q(b|t) \equiv 1 - q(a|t)$. In that case, the voter's utility is 1 if B wins, and remains equal to -1 if C wins. This is the second line of (1). In each case, the *expected* utility depends on the probability that each candidate wins the election. It is not yet the place to specify these probabilities, because they depend on the other voters' strategy.

An instrumental voter uses his ballot to maximize his expected utility (1). Let us focus on type t_A and t_B voters. Casting a ballot on, say, A increases the number of votes of A by 1. In state a, this **increases** their utility by 1 if the vote is pivotal against B, and by 2 if it is pivotal against C. If the state is b, then the voter's utility **decreases** by 1 if the vote is pivotal against B and

increases by 1 if pivotal against C. Thus, the *expected gain* G of a vote for A is:

$$\begin{aligned} G\left(A|t\right) &= q\left(a|t\right)\ \left\{\Pr\left(piv_{AB}|a\right)+2\Pr\left(piv_{AC}|a\right)\right\}+... \\ &\quad ...+q\left(b|t\right)\ \left\{\Pr\left(piv_{AC}|b\right)-\Pr\left(piv_{AB}|b\right)\right\},\ \text{ if } t\in\left\{t_A,t_B\right\}. \end{aligned}$$

The expected gain depends on the probability that a ballot is pivotal against either party. We denote for instance $\Pr\left(piv_{AB}|a\right)$ the probability that the vote for A is pivotal against B in state a. A priori, the voter thus has to solve a rather complex problem. Given the strategy of the other voters, he should evaluate the probability to be pivotal against each party in each state of nature, and for there compute the value of $G\left(A|t\right)$ and $G\left(B|t\right)$. However, we will see that, once we use the properties of Poisson Games highlighted by Myerson (2000), this apparently complex problem boils down to a much simpler one.

It is straightforward to see that types t_A and t_B never want to vote for C. Depending on which of $G\left(A|t\right)$ and $G\left(B|t\right)$ is the largest, they vote for A or B. To compare these two values, we use the difference:

$$\begin{aligned} G\left(A|t\right)-G\left(B|t\right) = &\ q\left(a|t\right)\ \left\{\Pr\left(piv_{AB}|a\right)+2\Pr\left(piv_{AC}|a\right)+\Pr\left(piv_{BA}|a\right)-\Pr\left(piv_{BC}|a\right)\right\} \\ &+q\left(b|t\right)\ \left\{\Pr\left(piv_{AC}|b\right)-\Pr\left(piv_{AB}|b\right)-\Pr\left(piv_{BA}|b\right)-2\Pr\left(piv_{BC}|b\right)\right\}, \end{aligned} \tag{2}$$

where piv_{PQ} denotes the event that the vote is pivotal between candidates P and Q. If that difference is positive, the voter casts his ballot on A. If it is negative, he votes for B. We shall see that, if B is expected to rank third, then all t_A and t_B voters unambiguously want to vote for A and conversely if A is expected to rank third.

Formally, we exploit Theorem 1 of Myerson (2000) to characterize these pivot probabilities in a Poisson Game. They centrally depend on the *magnitude,* denoted *mag*, of the considered event λ (for instance, piv_{AB} is the event that $\lambda_A=\lambda_B$):

Property 1 *(Myerson 2000, Theorem 1) Subject to* $\sum_P\lambda\left(P\right)=1$*,* $\sum_P\tau(P|\omega)=1$*,* $P\in\{A,B,C\}$*, and for* $\omega\in\{a,b\}$*, the probability that* ***realized*** *vote shares are* $\lambda=\left\{\lambda\left(A\right),\lambda\left(B\right),\lambda\left(C\right)\right\}$ *when* ***expected*** *vote shares are* $\tau(\omega)$ *is*:

$$\Pr(\lambda|\tau(\omega))\underset{n\longrightarrow\infty}{\longrightarrow}\max_{\lambda}\frac{\exp[mag\left[\lambda\right]\ \ n]}{\prod\limits_P\sqrt{2\pi n\lambda\left(P\right)+\frac{\pi}{3}}},$$

$$\textit{where:}\qquad mag\left[\lambda\right]=\sum_P-\lambda\left(P\right)\ \ \log(\frac{\lambda\left(P\right)}{\tau(P|\omega)})\ \ (\leq 0)$$

From Property 1, we see that the probability ratio between two events λ and λ' either converges to infinity or to zero if these two events have a different *magnitude*:

Property 2 *(Myerson 2000, Corollary 1) The relative probability of two events* λ *and* λ' *converges to 0 as population size increases to infinity when the magnitude of* λ *is larger than that of* λ'*, and* *conversely:*

$$\begin{aligned}\frac{\Pr(\lambda|\tau(\omega))}{\Pr(\lambda'|\tau(\omega))} \quad &\underset{n\longrightarrow\infty}{\longrightarrow} 0 \text{ \textit{if} } mag\,[\lambda] < mag\left[\lambda'\right] \leq 0 \\ &\underset{n\longrightarrow\infty}{\longrightarrow} \infty \text{ \textit{if} } mag\left[\lambda'\right] < mag\,[\lambda] \leq 0\end{aligned}$$

The usefulness of Property 2 stems from the fact that we will only need to compare the magnitudes of the pivot probabilities to understand how $G(A|t)$ and $G(B|t)$ compare to one another, and in this way know the sign of the difference (2). Indeed, by Property 2, if any one of the magnitudes in $\Pr(piv_{AB}|a)$, $\Pr(piv_{AC}|a)$, or $\Pr(piv_{AC}|b)$ is larger than that of $\Pr(piv_{AB}|b)$, $\Pr(piv_{BC}|a)$ and $\Pr(piv_{BC}|b)$, then a vote for A must have a larger value than a vote for B, independently of the type being t_A or t_B. Therefore, none of the majority-block voters wants to vote for the latter. The following Lemma establishes that this mechanism operates as soon as a party is anticipated to be third in the electoral race:

Lemma 1 *The magnitude of the pivot probability between two parties P and Q is:*

$$mag\,(piv_{PQ}|\omega) = \log\left(1-\left(\sqrt{\tau(P|\omega)}-\sqrt{\tau(Q|\omega)}\right)^2\right) \simeq -\left(\sqrt{\tau(P|\omega)}-\sqrt{\tau(Q|\omega)}\right)^2,$$

if these two parties are the top two candidates, and it is smaller than that value for the bottom two candidates. Hence, whenever the three parties have different expected vote shares, $\tau(P|\omega) > \tau(Q|\omega) > \tau(R|\omega)$, the pivot probability between P and Q has the largest magnitude. Conversely, that between Q and R has the smallest magnitude.

Proof. See appendix. ■

Therefore, a sufficient condition for a Duvergerian equilibrium is that either A or B is perceived as trailing behind the other two candidates in both states of nature. Duvergerian equilibria are therefore self-fulfilling in plurality elections: if voters expect that one of the two candidates is trailing behind, then no voter wants to waste his ballot on that candidate, who receives no vote. The situation is more intricate when these parties are trailing behind in only one state, or if they are leading the polls in both states. We analyze that situation in the next section.

3.2 A Condorcet equilibrium may not exist

The literature on the Condorcet jury theorem typically focuses on two-candidate elections. Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997) and Myerson (1998) for instance analyze a situation in which only candidates A and B exist. In the absence of candidate C, our setup is identical to the one of Myerson (1998). His Theorem 2 shows that a Condorcet equilibrium always exists in such a game. In that equilibrium, with a probability that approaches 1 as the size of the electorate increases, candidate A wins in state a and candidate B wins in state b. This equilibrium requires that voters adopt a strategy such that:

$$\begin{cases} \tau(A|a) > \tau(B|a), \\ \tau(B|b) > \tau(A|b). \end{cases} \tag{3}$$

Austen-Smith and Banks (1996) show that such an equilibrium also exists under a different distribution of voters, and Feddersen and Pesendorfer (1997) show that a property of *full information equivalence* holds, even if voters have heterogeneous preferences regarding candidates A and B: with a probability that converges to 1 as population size increases, the winning candidate is the same as if voters exactly knew the state of nature.

In our setup, matters become more complex than in Myerson (1998): a Condorcet-type of equilibrium, in which A wins in state a and B wins in state b would be Pareto superior to a Duvergerian equilibrium as described above. However, to exist, such an equilibrium would require that:

$$\begin{cases} \tau(A|a) > \max\left[\tau(B|a), \tau\left(C\right)\right], \\ \tau(B|b) > \max\left[\tau(A|b), \tau\left(C\right)\right]. \end{cases} \tag{4}$$

That is, it is no longer sufficient that A is above B is state a and conversely in state b. We also need that this leading party be above C as well.

Condorcet-like equilibria require that neither A nor B are deserted by the voters. Hence, the difference (2) can neither be strictly positive nor strictly negative both for type t_A and t_B at the same time. In light of Properties 1 and 2 above, a necessary condition is therefore that the magnitude of the pivot probabilities in favor of A and in favor of B be identical. Otherwise, voting for one of the parties would become a dominated action.

Our next proposition shows that there always exists a strategy function σ such that he pivot probabilities in favor of A and in favor of B are identical. However it is only "stable" if C does not win the election under that strategy function. "Stability" here is used in the same way as in a Cournot equilibrium: assume that expected vote shares are $\tau^0(P|\omega)$. Given $\tau^0(P|\omega)$, allow a tiny fraction of the electorate to choose their strategy, and then compute the new expected vote shares $\tau^1(P|\omega)$, and iterate to identify a sequence $\tau^k(P|\omega)$, $k = 1, 2, ...$ We call an equilibrium "stable" if there exists a neighbourhood of the equilibrium $\tau^*(P|\omega)$ such that the sequence $\tau^k(P|\omega)$ converges to $\tau^*(P|\omega)$.

Our second proposition shows that:

Proposition 2 *Under plurality, the only non-Duvergerian equilibrium is such that:*

$$\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0.$$

If and only if, in that point, we have $\tau(A|a) \simeq \tau(B|b) > \tau\left(C\right)$*, then this equilibrium is "stable" and Condorcet-like: for* $n \to \infty$*, the best candidate is elected with a probability that converges to*

1. If the opposite inequality holds, then this equilibrium is "unstable" and, with a probability that converges to 1 as $n \to \infty$, it elects dominated candidate C.

Proof. See Appendix. ■

This proposition extends Theorem 2 in Myerson (1998) to a three-candidate setting. Like in that theorem, there always exists an equilibrium satisfying (3). However, there might not be any equilibrium satisfying the more stringent condition (4). In the latter case, the equilibrium is not Condorcet-like: it is the worst candidate that is elected.

An important feature that is central for Condorcet-like equilibria throughout the entire paper is that the expected vote share of candidate A in state a is identical to that of candidate B in state b. This feature ensures that a vote for A is as likely to be pivotal in state a as a vote for B is in state b. This is a necessary condition for the difference (2) not to be always positive or negative.

3.3 Wrap up

This section has shown that, under plurality elections, two types of equilibria coexist. We call "Duvergerian" the equilibria in which only two candidates get votes. By definition, this excludes the possibility that candidate A wins in state a and candidate B wins in state b. There also exists another type of equilibrium, in which the three candidates get votes. This type of equilibrium requires that the magnitude of the pivot probabilities in favor of A in state a is the same as that in favor of B in state b. This is a necessary condition to ensure that majority-block voters do not overwhelmingly prefer one of the two candidates. This equilibrium can be "Condorcet-like", i.e such that the best candidate is elected with a probability that converges to 1, in which case it is stable. But, if types t_C are too numerous, then C wins with a probability that converges to 1. In that case, this equilibrium is inferior to a Duvergerian equilibrium, and it is unstable.

4 Runoff elections - `To be completed`

Runoff elections are often expected to settle the issue of Duvergerian equilibria. Piketty (2000) for instance claims that runoff elections should be able to separate the communication stage, in which voters learn which of A and B is best, from the election stage. Instead, we show that Duvergerian equilibria always exist also in runoff elections.

To this end, consider a two-round election in which a candidate wins immediately in the first round if he has more than 50% of the votes. If no candidate has more than 50% of the votes in the first round, then a runoff election is organized amongst the two candidates who obtained the most votes in the first round.

Clearly, like in plurality elections, types t_C always vote C in the first round, and types t_A and t_B never vote for C, either in the first or in the second round. Yet, if, say, B is expected to receive almost no votes in the first round, we must have: $\tau(A|\omega) > 1/2 > \tau(C|\omega) = r(t_C) > \tau(B|\omega) = \varepsilon$. Majority block voters then face the following trade-off: either they vote for A, to help him win in the first round, or they vote for B, in the hope that they are pivotal in obtaining a second round between A and B.

Our first theorem establishes that, face with such an expected strategy profile, majority-block voters prefer to vote for A. The reason is that, the joint probability that a vote for B is pivotal in bringing B above C *and* that A does not win in the first round is infinitesimally low compared to the probability that a vote for A is pivotal in making him win in the first round. This implies that the value of a vote for A is necessarily larger than the value of a vote for B, despite the fact that the risk of having C win the second is extremely low:

Theorem 1 *Like under plurality elections, run-off elections feature at least one inferior, Duvergerian, equilibrium in which the winning candidate is chosen independently of the state of nature, and the third candidate (A or B) receives zero votes.*

Proof. `To be written` ■

Note that this theorem does *not* claim that Condorcet-like equilibria do not exist under runoff elections. Runoff elections may actually feature many types of equilibria, some being Condorcet-like –see Bouton (2007) for a more extensive analysis of runoff elections.

5 Approval Voting

Under approval voting, voters can vote for as many (or as few) candidates as they wish. Each approval counts as one vote: when a voter only approves of 'A', candidate A is credited with one vote. If the voter approves of both A and B (we denote this action AB), then both A and B are credited with one vote. Eventually, the candidate with the most votes wins. Note that approving of all candidates is equivalent to abstaining in this game. We therefore do not need to introduce abstention in the voters' action set. Note also that an AB-approval cannot be pivotal between A and B.

Hence, the only change with respect to plurality elections is that the set of feasible actions is enlarged:[2]

$$\Psi = \{A, B, C, AB, AC, BC, ABC\}.$$

Majority block voters necessarily reduce their utility if their vote is pivotal in favor of C. Hence, C, AC and BC are dominated actions. By the same token, ABC is dominated by AB, since the

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[2] Under plurality elections, the voter only has access to the action set $\Psi = \{A, B, C\}$.

latter allows them to be pivotal against C.[3] This shows that, in equilibrium, their ballot will never include C, i.e.:

$$\sigma^* (\psi|t) > 0, \forall t \in \{t_A, t_B\} \implies \psi \in \{A, B, AB\}.$$

The central difference between plurality and approval elections is therefore that t_A and t_B voters are no longer constrained to choose between A and B. They can also approve of both A and B together. The main difference between the first two actions and the last one is that an AB-approval can only be pivotal against C, in contrast to the only A and only B approvals. Framed differently, if the voter is uncertain between whom of A and B should win the election, he can cast a ballot AB, which can only be pivotal against C. As we show now, this implies that, in equilibrium, voters mix between at most two actions, and never strictly mix between approving of A and approving of B.

Let $G(\psi|t)$ denote the expected payoff associated with action ψ for a voter with type t. For $t \in \{t_A, t_B\}$, we have:

$$\begin{aligned} G(A|t) = \quad & q(a|t) \cdot [\Pr(piv_{AB}|a) + 2\Pr(piv_{AC}|a)] \\ & + q(b|t) \cdot [-\Pr(piv_{AB}|b) + \Pr(piv_{AC}|b)], \end{aligned} \tag{5}$$

$$\begin{aligned} G(B|t) = \quad & q(a|t) \cdot [-\Pr(piv_{BA}|a) + \Pr(piv_{BC}|a)] \\ & + q(b|t) \cdot [\Pr(piv_{BA}|b) + 2\Pr(piv_{BC}|b)], \end{aligned} \tag{6}$$

and

$$\begin{aligned} G(AB|t) = \quad & q(a|t) \cdot [\Pr(piv_{BC}|a) + 2\Pr(piv_{AC}|a)] \\ & + q(b|t) \cdot [\Pr(piv_{AC}|b) + 2\Pr(piv_{BC}|b)]. \end{aligned} \tag{7}$$

The following lemma establishes that, in a mixed strategy equilibrium, t_A-voters will never choose action B, because it is dominated by AB. Similarly, action AB dominates action A for t_B-voters, and types t_C only approve of C:

Lemma 2 *In a mixed-strategy equilibrium, majority-block voters always adopt a strategy such that:* $\sigma(A|t_A) + \sigma(AB|t_A) = 1$ *and* $\sigma(B|t_B) + \sigma(AB|t_B) = 1$*. That is, they only mix between their 'own candidate' and jointly approving of A and B. Conversely, minority-block voters, t_C, always vote for C with probability 1.*

Proof. See appendix ■

The intuition for this lemma is that a voter would only mix between actions A and B if he were indifferent between the two candidates. Indifference however means that the voter does not want to choose between these candidates. Therefore, a safer option is to choose action AB, which is pivotal against C with a higher probability, but cannot be mistakenly pivotal, e.g. in favour of

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[3] Hence, abstention would also be a dominated action for types t_A and t_B if it were part of their action set. The same holds for types t_C who always benefit from being pivotal in favour of C.

A against B when the true state of nature is b. To establish that AB dominates the other two actions in the case of indifference, the proof of the lemma first considers the necessary condition for indifference between A and B: the perceived probability to be in either of the two states of nature must exactly match the weighted sum of the pivot probabilities in each state. Approving of A dominates AB only if the probability to be in state a is sufficiently large, and approving of B dominates AB only if that probability is sufficiently low. Yet, we demonstrate that, at the point of indifference between A and B, the probability to be in state a is too low for the former to hold, and too large for the latter. Taking account of the difference between t_A and t_B voters, we then show that, in equilibrium, the former type must mix between A and AB approvals, whereas the latter type must mix between B and AB approvals.

The first implication of this lemma is that only four actions can be used in equilibrium: A, B, C, and AB. By the definition of approval voting, the number of votes in favour of A is then the sum of the number x of voters who choose action A and that who choose action AB, and similarly for candidate B. Hence, A wins the election if:

$$x\left(A\right)+x\left(AB\right)\geq\max\left[x\left(B\right)+x\left(AB\right),\ x\left(C\right)\right],$$

and conversely for the candidates B and C.

The dominance of AB over a mixture between A and B, established in Lemma 2, will be of prime importance to establish that:

Theorem 2 *Under approval voting, the voting equilibrium is unique and of the 'Condorcet type': candidate A (respectively B) is elected in state a (respectively b) with a probability that approaches 1 as population size becomes infinitely large. The voting strategies supporting the equilibrium feature:*

$$\begin{aligned}&\sigma\left(A|t_A\right)=\sigma\left(B|t_B\right)\left(\frac{\sqrt{r(t_B|a)}+\sqrt{r(t_B|b)}}{\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}}\right)^2; &&\sigma\left(AB|t_A\right)=1-\sigma\left(A|t_A\right);\\ &\sigma\left(AB|t_B\right)=1-\sigma\left(B|t_B\right); &&\textit{and } \sigma\left(C|t_C\right)=1,\end{aligned}$$

and they imply the expected ranking:

$$\begin{aligned}&\tau(A|a)+\tau(AB|a)>\tau(B|a)+\tau(AB|a)>\tau\left(C\right) &&\textit{in state } a, \textit{ and}\\ &\tau(B|b)+\tau(AB|b)>\tau(A|b)+\tau(AB|b)>\tau\left(C\right) &&\textit{in state } b.\end{aligned}$$

Proving this theorem is the purpose of the remainder of this section. The first step will be to identify the pivot probabilities involving candidate C and to show that, whenever C is expected to be one of the top-two candidates, their magnitude always dominate the magnitude of the pivot probability between A and B (Lemma 3).Then, we identify a necessary condition for an equilibrium: at least one of the two types must be indifferent between a single-A or a single-B approval, and a joint AB-approval, and no type can strictly prefer an AB-approval (Proposition 3). This excludes the possibility that C ranks first or second: there cannot be a Duvergerian equilibrium under approval voting. Nest, we show in Proposition 4 that a Condorcet-like equilibrium must exist, and in Lemma 4 that this is the onllu possible equilibirum under approval voting.

Together, these two Lemma and one Proposition shape the proof of Theorem 2.

Under plurality and runoff elections, the number of votes collected by each candidate are distributed according to independent Poisson distributions, with a mean and a variance that are equal to the expected number of voters who cast their ballot for that party. Property 2 in Section 3 establishes that, in this case, the expected ranking of the candidates translates one for one in a given ranking of the magnitudes regarding the probability that a vote is pivotal between two candidates. That is, when population size increases to infinity, a vote becomes infinitely more likely to be pivotal among the top two candidates than among any other pair of candidates.

Under approval voting, the situation becomes different. While some voters keep approving of only one candidate, there may be a positive fraction of voters who jointly approve of candidates A and B. Therefore, the total number of votes in favour of candidate A $(x(A) + x(AB))$ is no longer statistically independent of the number of votes in favour of candidate B $(x(B) + x(AB))$. Yet, the number of AB-approvals remains statistically independent of the number of A, B and C single approvals. Clearly, the last three are also statistically independent of one another, like under plurality or run-off elections. Therefore, we can still apply Theorem 1 in Myerson (2000) (see Property 1 in Section 3), provided that we make a distinction between each type of *strategy* that is played with positive probability in equilibrium, and not between the total number of *votes* attributed to each candidate.

An implication of this distinction is that, unlike in Property 2, the expected ranking of the candidates need not translate one for one in a given ranking of the pivot probabilities. Indeed, when we focus on the probability that a vote is pivotal between A and B, we only need to take account of the expected number of voters who choose action A and action B. The reason is that AB-approvals cannot be pivotal among these candidates. Instead, the probability that a vote is pivotal between A (or B) and C does depend on the expected number of voters who choose action AB. Therefore, the variance of the distributions of votes differs according to the candidates considered. The following lemma establishes that, when C is expected to rank first or second, the mapping of the expected vote ranking into the magnitude ranking still holds. Note that the Lemma makes no mention of what happens when C is expected to rank third. This is because, when C is expected to rank third, the magnitude of the pivot probability between the first and third candidates can still be larger than the one between the top two candidates. We return to this point later:

Lemma 3 *If C is expected to rank first or second in state ω, then, for $\tau(A|\omega) > \tau(B|\omega)$:*

$$mag\left(piv_{AC}|\omega\right) = \log\left(1 - \left(\sqrt{\tau\left(A|\omega\right) + \tau\left(AB|\omega\right)} - \sqrt{\tau\left(C|\omega\right)}\right)^2\right),$$

and we have:

$$mag(piv_{AC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega).$$

Conversely, for $\tau(A|\omega) < \tau(B|\omega)$, we have $mag(piv_{BC}|\omega) > mag(piv_{AB}|\omega) \geq mag(piv_{AC}|\omega)$. That is, whenever C is expected to rank first or second, the pivot probability between the expected top *(resp.* bottom*) two candidates has the largest (resp. smallest) magnitude.*

Proof. See appendix ■

This lemma implies that, for large populations, pivot probabilities involving candidate C become infinitely more likely than the one between A and B. However, it turns out that, whenever this holds, the value of an AB-approval becomes infinitely larger than a single A or a single B approval for both types t_A and t_B. This means that, upon the expectation that C might rank first or second in either state of nature, all types t_A and t_B would cast an AB-approval. Since, together, these two types are a majority of the population, both A and B would come ahead of C. In other words, we can establish that:

Proposition 3 *(No Duvergerian equilibrium)Under approval voting, a necessary condition to have an equilibrium is:*

$$\min\left\{G(A|t_A) - G(AB|t_A), G(B|t_B) - G(AB|t_B)\right\} = 0,$$

which implies that, in equilibrium, C must be expected to rank third in both states of nature. Hence, a Duvergerian equilibrium cannot exist under approval voting.

Proof. See appendix ■

This proposition shows that, to have an equilibrium, at least one of the majority-group types, t_A or t_B, must be indifferent between casting an AB-approval and voting for their *a priori* preferred candidate, A or B. This indifference excludes two types of equilibria. First, Duvergerian equilibria are excluded: if either A or B is expected to trail behind, then both types t_A and t_B prefer to jointly approve of A and B. In contrast to the forces that induce voters to abandon the trailing candidate under either plurality or run-off elections, supporting the trailing candidate cannot be costly, since the voters can, at the same time, keep approving of the leading candidate.

Second, this indifference also excludes the potential equilibrium in which all voters in the majority group would jointly approve of A and B. This contrasts with Nagel (2007), who sustains that such equilibria, which he calls the 'Burr Dilemma' of approval voting, are the main trap in approval voting. His argument is that approval voting is biased towards ties among the leaders:

"[The approval voting] *experiment ended disastrously in 1800 with the infamous Electoral College tie between Jefferson and Burr*".(Nagel 2007). In contrast, Proposition 3 demonstrates that this cannot hold in equilibrium, when voters adopt non-dominated strategies. We therefore conclude that the 'Burr Dilemma' could only exist because of the many deadlocks that appeared at that time, which constrained the Federalists and the Republicans to impose particular voting instructions to their members. These members were clearly tempted to deviate from this instructed strategy:

> [Burr] was willing only "if assurances can be given that the southern states will act fairly." Jefferson's emissary, Albert Gallatin, pledged that they would (Wills 2003, 71). Soon afterwards, a meeting of Republicans in Philadelphia duly endorsed a ticket of Jefferson and Burr. Any breach of the bargain would tarnish the personal honor of leaders on both sides, given the strong commitments they had made.[... As Madison] noted, confidence (conveyed by Burr's emissaries) that New York would hold for Jefferson impelled Virginia "to give an unanimous tho' reluctant vote for B. as well as J."

This shows that the tie that occurred in 1800 was clearly not a best response for each individual voter, and that it could not have been sustained as an equilibrium in a large electorate. Therefore, while Nagel's argument is clearly valid for small electorates in which individual actions can be in dominated strategies, it does not extend to larger electorates in which voters play non-cooperatively.

To the contrary, in our model and provided that an equilibrium exists, any equilibrium must be Condorcet-like:

Proposition 4 *(Existence of a Condorcet Equilibrium) There always exists a pair of strategies* $\sigma(A|t_A)$ *and* $\sigma(B|t_B)$ *such that:*

$$\sigma(A|t_A) = \sigma(B|t_B) \times \left(\frac{\sqrt{r(t_B|a)}+\sqrt{r(t_B|b)}}{\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}}\right)^2 = \sigma(B|t_B)\ \ \rho,$$

which are a stable equilibrium of the approval voting game and which ensure that the magnitudes $mag(piv_{AB}|a)$ *and* $mag(piv_{AB}|b)$ *are equal and at least as large as any other magnitude. This equilibrium is Condorcet-like and elects candidate A (respectively B) in state a (resp. b) with a probability that converges to 1 as* $n \to \infty$*.*

Proof. See appendix ■

It remains to prove unicity.

For $\sigma(A|t_A)$ equal to $\sigma(B|t_B)\,\rho$, it is straightforward from the proof of Proposition 4 that there is only one value of $\sigma(B|t_B)$ that can be an equilibrium. Indeed, since the magnitudes of the pivot probabilities are monotonically decreasing in the gap between the vote shares of the parties, the lower is $\sigma\left(B|t_B\right)$, the larger is the magnitude $mag\left(piv_{AB}|\omega\right)$ and, when A and B are leading in their respective state, the smaller are the magnitudes involving party C. Therefore, there can be either one or zero interior solution $0 < \sigma\left(B|t_B\right) < 1$ ensuring that the condition for indifference $G(B|t_B) = G(AB|t_B)$ holds. As seen in the proof of the proposition, if there is no such interior solution, then $\sigma\left(B|t_B\right) = 1$ is the (corner) solution.

Proving unicity therefore only requires to check that $\sigma(A|t_A)= \sigma\left(B|t_B\right)\rho$ is indeed the only possible strategy for types t_A, given the strategy $\sigma\left(B|t_B\right)$. This is the purpose of the following lemma:

Lemma 4 *(Unicity) The strategy*

$$\sigma\left(A|t_A\right) = \sigma\left(B|t_B\right)\frac{\left(\sqrt{r\left(t_B|a\right)}+\sqrt{r\left(t_B|b\right)}\right)^2}{\left(\sqrt{r\left(t_A|a\right)}+\sqrt{r\left(t_A|b\right)}\right)^2}; \quad \sigma\left(AB|t_A\right) = 1 - \sigma\left(A|t_A\right)$$

is the only possible best response of types t_A for a given mixture by types t_B.

Proof. See appendix ■

This unicity result contrasts with De Sinopoli et al. (forthcoming): in a model where voters have known and fixed preferences, they show that approval voting also features dominated equilibria. We therefore conclude that, if cleavages are too strong and too few voters are willing to change their mind regarding which candidate best fits their needs, then none of the electoral systems considered here clearly outperforms the others. If instead undecided/uncertain voters are sufficiently numerous in the election, then approval voting unambiguously dominates the plurality and runoff electoral systems.

6 Extensions and Discussion

To be written

7 Conclusion

To be written

8 Appendix

Proof of Lemma 1

Proof. First, we use Property 1 to compute the magnitude of the probability that P and Q have the same vote share in the state ω. That is, $mag(piv_{PQ}|\omega)$ is the maximum of $mag\left[\lambda\right]$ subject to $\lambda_P = \lambda_Q = \lambda$:

$$\begin{aligned} mag(piv_{PQ}|\omega) &= \max_{\lambda} 2\lambda + (1-2\lambda) - 1 \\ &\quad - \log\left[\left(\frac{\lambda}{\tau(P|\omega)}\right)^{\lambda}\left(\frac{\lambda}{\tau(Q|\omega)}\right)^{\lambda}\left(\frac{1-2\lambda}{\tau(R|\omega)}\right)^{1-2\lambda}\right] \\ &= \max_{\lambda} \log\left[\left(\frac{\tau(P|\omega)}{\lambda}\right)^{\lambda}\left(\frac{\tau(Q|\omega)}{\lambda}\right)^{\lambda}\left(\frac{\tau(R|\omega)}{1-2\lambda}\right)^{1-2\lambda}\right]. \end{aligned}$$

Taking first and second order conditions, we find that this is maximized in

$$\lambda^{**}_{PQ}\left(\omega\right) = \frac{\sqrt{\tau(P|\omega)\tau(Q|\omega)}}{2\sqrt{\tau(P|\omega)\tau(Q|\omega)} + \tau(R|\omega)}.$$

Second, a vote can only be pivotal between P and Q if the third candidate, R, has fewer votes than P and Q. This imposes an additional condition: $\lambda \geq 1-2\lambda$. Introducing that condition in the maximization problem, we find

$$\lambda^{*}_{PQ}\left(\omega\right) = \max\left\{\lambda^{**}_{PQ}\left(\omega\right), 1/3\right\}$$

(Note that $mag(piv_{PR}|\omega)$ and $mag(piv_{QR}|\omega)$ can be computed in the same way). Whenever $\lambda^{*}_{PQ}\left(\omega\right) = \lambda^{**}_{PQ}\left(\omega\right)$, $mag(piv_{PQ}|\omega)$ becomes:

$$mag\left(piv_{PQ}|\omega\right) = \log\left(1 - \left(\sqrt{\tau\left(P|\omega\right)} - \sqrt{\tau\left(Q|\omega\right)}\right)^2\right)$$

while if $\lambda^{*}_{PQ}\left(\omega\right) = 1/3$, the magnitude becomes:

$$mag(piv_{PQ}|\omega) = \log(3[\tau\left(P|\omega\right)\tau\left(Q|\omega\right)\tau\left(R|\omega\right)]^{1/3})$$

Since $\tau(P|\omega) > \tau(Q|\omega) > \tau(R|\omega)$, we have that $\lambda^{**}_{PQ}\left(\omega\right) > 1/3$, and that $\lambda^{**}_{QR}\left(\omega\right) < 1/3$. Therefore, $\lambda^{*}_{PQ}\left(\omega\right) = \lambda^{**}_{PQ}\left(\omega\right)$, and $\lambda^{*}_{QR}\left(\omega\right) = 1/3$, which implies that:

$$\begin{aligned} mag\left(piv_{PQ}|\omega\right) &= \log\left(1 - \left(\sqrt{\tau\left(P|\omega\right)} - \sqrt{\tau\left(Q|\omega\right)}\right)^2\right) \\ mag(piv_{QR}|\omega) &= \log\left(3[\tau\left(P|\omega\right)\tau\left(Q|\omega\right)\tau\left(R|\omega\right)]^{1/3}\right) \end{aligned}$$

Concerning $mag(piv_{PR}|\omega)$, the situation is a priori unclear since $\lambda^{**}_{PR}\left(\omega\right) \in (0, 1/2)$. Nonetheless, $\left(\sqrt{\tau\left(P|\omega\right)} - \sqrt{\tau\left(Q|\omega\right)}\right)^2 < \left(\sqrt{\tau\left(P|\omega\right)} - \sqrt{\tau\left(R|\omega\right)}\right)^2$ implies that $mag(piv_{PQ}|\omega) > mag(piv_{PR}|\omega)$ when $\lambda^{*}_{PR}\left(\omega\right) = \lambda^{**}_{PR}\left(\omega\right)$. Since $\lambda^{*}_{PR}\left(\omega\right) = 1/3$ results from an additional constraint, it is obvious that $mag(piv_{PR}|\omega)$ computed for this value of $\lambda^{*}_{PR}\left(\omega\right)$ is smaller or equal to the unrestricted value, that is when computed in $\lambda^{*}_{PR}\left(\omega\right) = \lambda^{**}_{PR}\left(\omega\right)$. This reinforces the inequality.

Finally, since $mag(piv_{QR}|\omega)$ is identical to the restricted magnitude of $mag(piv_{PR}|\omega)$, it follows directly that $mag\left(piv_{PQ}|\omega\right) > mag(piv_{PR}|\omega) \geq mag(piv_{QR}|\omega)$. ■

Proof of Proposition 2

Proof. We first prove that such an equilibrium exists and necessarily entails $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$. That is, there is a *unique* non-Duvergerian equilibrium. Second, we verify whether it satisfies the "stability" properties.

Step 1. $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$ is an equilibrium

Following Theorem 2 of Myerson 1998, note that if a type $t \in \{t_A, t_B\}$ adopts a strictly mixed strategy, then the other type $t' \neq t$, $t' \in \{t_A, t_B\}$ votes for "his" candidate with probability 1. The reason is that the priors $q(a|t)$ and $q(b|t)$ are different across types, which implies $G(A|t_A) - G(B|t_A) > G(A|t_B) - G(B|t_B)$ for any expected voting profile.

Having noted this, we know that this strategy profile is an equilibrium if, for that ranking of the vote shares, we have:

$$\begin{aligned} &G(A|t_A) - G(B|t_A) \geq 0, \text{ and} \\ &G(A|t_B) - G(B|t_B) \leq 0, \end{aligned} \tag{8}$$

with at least one strict inequality. That is, types t_A must be willing to support A, and conversely for types t_B. Using (2), it is immediate to check that these inequalities hold iff $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$

Next, remark that: *a)* pivot probabilities are continuous in the voters' propensity to cast their ballot on A and on B, and *b)* payoffs are bounded. Therefore, the difference $G(A|t) - G(B|t)$ is continuous in the voters' propensity to vote for A, and we can apply Kakutani's fixed point theorem.

If voters marginally increase their propensity to vote A above the point in which $\tau(A|a) \simeq \tau(B|b)$, we have: $\tau(A|a) > \tau(B|b) > \tau(A|b) > \tau(B|a)$ and, as we show in steps 2a and 2b below:

$$\begin{aligned} G(A|t) - G(B|t) &> 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(A|a) \simeq \tau(B|b) > \tau(C) \\ G(A|t) - G(B|t) &< 0 \text{ for both } t \in \{t_A, t_B\}, \text{ if } \tau(A|a) \simeq \tau(B|b) < \tau(C), \end{aligned}$$

and the inequalities are reversed if the voters' propensity to vote for B increases. Two conclusions follow: *i) existence:* there must exist a strategy profile in the neighborhood of $\tau(A|a) = \tau(B|b)$ such that (8) holds. *ii) uniqueness:* $\tau(A|a) \simeq \tau(B|b) > \tau(A|b) \simeq \tau(B|a) > 0$ is a necessary condition for the non-Duvergerian equilibrium.

Step 2a. If $\tau(A|a) \simeq \tau(B|b) > \tau(C)$, the equilibrium is stable

Generically, two sub-cases must be considered: $\tau(C) > \tau(A|b)$ and $\tau(C) < \tau(A|b)$.

If $\tau(C) > \tau(A|b)$ then the pivot probabilities between A and C in state a and between B and C in state B become infinitely larger than any other pivot probability, by Property 2. Hence, (2) converges to:

$$\lim_{\substack{n\to\infty \\ \tau(A|b)>\tau(C)>\tau(A|b)}} G(A|t) - G(B|t) = q(a|t)\ 2\Pr(piv_{AC}|a) - q(b|t)\ 2\Pr(piv_{BC}|b). \tag{9}$$

If the vote share of A increases (the argument is symmetric if it decreases), then $\tau(A|a) - \tau(C)$ increases, and $\tau(B|b) - \tau(C)$ decreases. This decreases the magnitude of the pivot probability in state a and increases it in state b. This implies that $G(A|t) - G(B|t)$ becomes negative. The equilibrium is thus "stable" in the sense that an exogenous increase in the vote share of A induces voters to increase their propensity to vote for B.

If $\tau(C) < \tau(A|b) \simeq \tau(B|a)$, then (2) converges to:

$$\begin{aligned}\lim_{\substack{n\to\infty \\ \tau(C)>\tau(A|b)\simeq\tau(B|a)}} G(A|t) - G(B|t) &= q(a|t)\ \ \{\Pr(piv_{AB}|a) + \Pr(piv_{BA}|a)\} \\ &\quad -q(b|t)\ \ \{\Pr(piv_{AB}|b) + \Pr(piv_{BA}|b)\}, \end{aligned} \tag{10}$$

and the same reasoning applies: an increase in the vote share of A implies $\Pr(piv_{AB}|a) + \Pr(piv_{BA}|a) << \Pr(piv_{AB}|b) + \Pr(piv_{BA}|b)$, which in turn implies that $G(A|t) - G(B|t)$ becomes negative for both t_A and t_B: the voter's propensity to vote for A has decreased.

Step 2b. If $\tau(A|a) \simeq \tau(B|b) < \tau(C)$, the equilibrium is unstable

If C has the largest vote share, (9) depicts again the value of a vote for A above that of B. However, the effect of an increase in the vote share of A is now reversed: $|\tau(A|a) - \tau(C)|$ decreases, and $|\tau(B|b) - \tau(C)|$ increases. This increases the magnitude of $\Pr(piv_{AC}|a)$ and decreases that of $\Pr(piv_{BC}|b)$. Hence, the initial rise in $\tau(A|a)$ induces the value of a vote for A to increase, for both types t_A and t_B. ■

Proof of Lemma 2

Proof. The fact that type t_C voters only vote for C is obvious. Turning to majority-block voters, since they only play A, B or AB with positive probability, only the number of ballots 'A' and 'B' matter to determine the pivot probabilities between A and B. Hence, the properties developed in the previous sections still apply.

We need to show that $\sigma(A|t) > 0$ implies $\sigma(B|t) = 0$ and conversely. We use a proof by contradiction.

First, note that a necessary condition for A and B being played with positive probability in equilibrium is that, for some $t \in \{t_A, t_B\}$:

$$G(A|t) = G(B|t) \geq G(AB|t), \tag{11}$$

Using (5) and (6), a necessary condition for $G(A|t) = G(B|t)$ is that:

$$\frac{q(a|t)}{q(b|t)} = \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b) + \Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a) + \Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)}. \tag{12}$$

Next, note that:

$$\begin{aligned} G(A|t) &\geq G(AB|t) \Longleftrightarrow \frac{q(a|t)}{q(b|t)} \geq M_1 \equiv \frac{\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)} \text{ if } \Pr(piv_{AB}|a) > \Pr(piv_{BC}|a) \\ G(B|t) &\geq G(AB|t) \Longleftrightarrow \frac{q(a|t)}{q(b|t)} \leq M_2 \equiv \frac{\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)} \text{ if } \Pr(piv_{BA}|b) > \Pr(piv_{AC}|b), \end{aligned}$$

and the reverse inequalities if $\Pr(piv_{AB}|a) < \Pr(piv_{BC}|a)$ and/or $\Pr(piv_{BA}|b) < \Pr(piv_{AC}|b)$.

Now, we prove the contradiction, i.e. that (11) can never hold: we first identify a lower bound for M_1 and an upper bound for M_2. Then, we show that this lower bound for M_1 is strictly larger than the upper bound for M_2, which contradicts the above condition since, to hold, condition (11) requires that:

$$M_1 \leq M_2. \tag{13}$$

M_1 is strictly increasing in $\Pr\left(piv_{BC}|\omega\right)$. Hence, a lower bound to M_1 is found by setting these two pivot probabilities equal to 0. Similarly, M_2 is strictly decreasing in $\Pr\left(piv_{AC}|\omega\right)$. An upper bound to M_2 is thus found by setting these equal to zero as well. This establishes that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{AB}|a\right)} < M_1 \text{ and } M_2 < \frac{\Pr\left(piv_{BA}|b\right)}{\Pr\left(piv_{BA}|a\right)},$$

and hence that a necessary condition for (13) is that:

$$\frac{\Pr\left(piv_{AB}|b\right)}{\Pr\left(piv_{BA}|b\right)}\frac{\Pr\left(piv_{BA}|a\right)}{\Pr\left(piv_{AB}|a\right)} < 1.$$

Using the definition of pivot probabilities in Poisson games and Myerson's (2000) offset theorem, the left-hand side of this expression is equal to:

$$\sqrt{\frac{\tau\left(B|b\right)}{\tau\left(B|a\right)}\frac{\tau\left(A|a\right)}{\tau\left(A|b\right)}},$$

which cannot be smaller than 1. Indeed, since $\frac{q(a|t_A)}{q(b|t_A)} > \frac{q(a|t_B)}{q(b|t_B)}$, types t_A necessarily vote for A with a probability higher than types t_B, by (12). Hence, in equilibrium, the vote share of A cannot be smaller in state a than in state b, and conversely for party B.

It follows that $G(A|t) = G(B|t)$ implies $G\left(AB, t\right) > G(A|t)$, and therefore that a strict mixture between A and B is a strictly dominated strategy. Finally, note that:

i) $G(A|t_A) = G\left(AB|t_A\right) \Rightarrow G(A|t_B) < G\left(AB|t_B\right)$ and $G(B|t_B) = G\left(AB, t_B\right) \Rightarrow G(B|t_A) < G\left(AB|t_A\right)$;
ii) $G(B|t_A) = G\left(AB, t_A\right) \Rightarrow \sigma\left(B|t_B\right) = 1$, which in turn implies that M_2 cannot be positive, by Lemma 1, and hence that $G(B|t_A) = G\left(AB, t_A\right)$ cannot be part of an equilibrium. Clearly, the same reasoning can be applied to show that $G(A|t_B) = G\left(AB, t_B\right)$ cannot be part of an equilibrium either. ■

Proof of Lemma 3

Proof. Let $\lambda(\psi)$ denote the realized share of voters who choose action $\psi \in \{A, B, C, AB\}$ and $\tau(\psi|\omega)$ denote the expected share of voters who choose action ψ in state ω. By the properties of Poisson Games, the different $\lambda(\psi)$ are independently distributed. We can thus use Property 1 to compute the magnitude of the probability that A and C have the same vote shares, which must be larger than that of B, in state ω:

$$\begin{aligned} mag(piv_{AC}|\omega) &= \max_{\lambda}\sum_{\psi}\lambda\left(\psi\right)\left(\log\ \tau\left(\psi|\omega\right) - \log\ \lambda\left(\psi\right)\right) \\ & \qquad s.t. \begin{cases} \lambda\left(A\right) + \lambda\left(AB\right) = \lambda\left(C\right) > \lambda\left(B\right) + \lambda\left(AB\right) \\ \sum\lambda\left(\psi\right) = 1. \end{cases} \end{aligned}$$

If we denote $\lambda\left(A\right) + \lambda\left(AB\right) = \lambda$, $\lambda\left(A\right) = \alpha\lambda$, and $\lambda\left(AB\right) = (1-\alpha)\lambda$, we can rewrite the magnitude of the pivot probability as:

$$\max_{\lambda}\sum_{\psi}\lambda\left(\psi\right)\left(\log\ \tau\left(\psi|\omega\right) - \log\ \lambda\left(\psi\right)\right) \qquad (14)$$

$$s.t. \begin{cases} \lambda\left(A\right) + \lambda\left(AB\right) = \lambda = \lambda\left(C\right) \\ \lambda\left(B\right) = 1 - 2\lambda < \alpha\lambda. \end{cases}$$

Abstracting from the condition $1-2\lambda<\alpha\lambda$, we find that this is maximized in:

$$\alpha_{AC}^{**}=\frac{\tau\left(A|\omega\right)}{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)} \tag{15}$$

$$\lambda_{AC}^{**}=\frac{\sqrt{\tau\left(C\right)\left[\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)\right]}}{2\sqrt{\tau\left(C\right)\left[\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)\right]}+\tau\left(B|\omega\right)} \tag{16}$$

Substituting for α_{AC}^{**} and λ_{AC}^{**} in (14) thus yields:

$$mag\left(piv_{AC}^{**}|\omega\right)=\log\left(1-\left(\sqrt{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(C|\omega\right)}\right)^{2}\right),$$

if the constraint $1-2\lambda<\alpha\lambda$ is not binding. If that constraint is instead binding, the magnitude of that probability would be some value $mag\left(piv_{AC}^{*}|\omega\right)$ which must be below $mag\left(piv_{AC}^{**}|\omega\right)$.

By analogy, denote $mag(piv_{BC}^{**}|\omega)$ the magnitude of the probability that B and C have the same vote share in the state ω and the equivalent constraint is not binding, and $mag(piv_{BC}^{*}|\omega)$ when it binds; and denote $mag(piv_{AB}^{**}|\omega)$ and $mag(piv_{AB}^{*}|\omega)$ the same probabilities regarding A and B. It is immediate to check that:

$$mag\left(piv_{BC}^{*}|\omega\right)\leq mag\left(piv_{BC}^{**}|\omega\right)=\log\left(1-\left(\sqrt{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(C|\omega\right)}\right)^{2}\right),$$

and

$$mag\left(piv_{AB}^{*}|\omega\right)\leq mag\left(piv_{AB}^{**}|\omega\right)=\log\left(1-\left(\sqrt{\tau\left(A|\omega\right)}-\sqrt{\tau\left(B|\omega\right)}\right)^{2}\right). \tag{17}$$

Now, note that, if the constraint is binding, we have: $\lambda\left(A\right)+\lambda\left(AB\right)=\lambda\left(C\right)=\lambda\left(B\right)+\lambda\left(AB\right)$, and the three events: piv_{AB}^{*}, piv_{AC}^{*} and piv_{BC}^{*} and therefore identical, which implies that:

$$mag\left(piv_{AC}^{*}|\omega\right)=mag\left(piv_{BC}^{*}|\omega\right)=mag\left(piv_{AB}^{*}|\omega\right). \tag{18}$$

We refer to these as *restricted magnitudes*.

Having observed this, we are now in a position to prove that, when the expected ranking is $A>C>B$ in state ω, then:

$$mag(piv_{AC}|\omega)>mag(piv_{AB}|\omega)\geq mag(piv_{BC}|\omega). \tag{19}$$

Since the proof proceeds in the same way for all the other possible rankings: $C>B>A$, $C>A>B$ and $B>C>A$, we shall not develop these.

The proof is in 3 steps: first, we compare the *unrestricted* magnitudes and show that: $mag(piv_{AC}^{**}|\omega)>mag(piv_{AB}^{**}|\omega)$. This amounts to showing that:

$$\tau(A|\omega)+\tau\left(AB|\omega\right)>\tau(C|\omega)>\tau(B|\omega)+\tau\left(AB|\omega\right) \tag{20}$$

implies:

$$\log\left(1-\left(\sqrt{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(C|\omega\right)}\right)^{2}\right)>\log\left(1-\left(\sqrt{\tau\left(A|\omega\right)}-\sqrt{\tau\left(B|\omega\right)}\right)^{2}\right).$$

Taking exponentials and rearranging terms, we find that this inequality holds iff:

$$\sqrt{\tau\left(A|\omega\right)}-\sqrt{\tau\left(B|\omega\right)}>\sqrt{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(C|\omega\right)}.$$

Adding and substracting $\sqrt{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}$ on the RHS of this inequality yields:

$$\begin{aligned}\sqrt{\tau\left(A|\omega\right)}-\sqrt{\tau\left(B|\omega\right)} \quad > \quad & \left(\sqrt{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}\right)+... \\ & ...+\left(\sqrt{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}-\sqrt{\tau\left(C|\omega\right)}\right),\end{aligned}$$

in which the LHS is larger than the first term of the RHS, by the concavity of the square root function, and the second term is negative, by the second inequality in (20). Hence, the ranking (20) indeed implies that: $mag(piv_{AC}^{**}|\omega) > mag(piv_{AB}^{**}|\omega)$.

Second, we show that $mag(piv_{AC}|\omega)$ is always equal to the unrestricted magnitude $mag(piv_{AC}^{**}|\omega)$. For this, we need to prove that:

$$\lambda\left(A\right)+\lambda\left(AB\right)=\lambda\left(C\right)>\lambda\left(B\right)+\lambda\left(AB\right) \tag{21}$$

at the optimum, that is:

$$\lambda_{AC}^{**}>1-2\lambda_{AC}^{**}+\left(1-\alpha_{AC}^{**}\right)\lambda_{AC}^{**}\Longleftrightarrow\lambda_{AC}^{**}>1/(2+\alpha_{AC}^{**}).$$

Using (15) and (16), and performing some manipulations, we see that the latter inequality holds iff:

$$\sqrt{\frac{\tau\left(C\right)}{\tau\left(A|\omega\right)+\tau\left(AB|\omega\right)}}>\frac{\tau\left(B|\omega\right)}{\tau\left(A|\omega\right)}, \tag{22}$$

in which both fractions are smaller than one. The latter implies that: $\frac{\tau(B|\omega)}{\tau(A|\omega)}\leq\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}\leq\sqrt{\frac{\tau(B|\omega)+\tau(AB|\omega)}{\tau(A|\omega)+\tau(AB|\omega)}}$, and by (20), the last member of this inequality is always smaller than $\sqrt{\frac{\tau(C)}{\tau(A|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{AC}|\omega)$ is always unrestricted. Hence $mag(piv_{AC}|\omega)$ is always larger than $mag(piv_{AB}|\omega)$, be the latter restricted or not.

Third, to complete the proof that (19) always holds under the expected ranking (20), it remains to demonstrate that $mag(piv_{AB}|\omega) \geq mag(piv_{BC}|\omega)$. To this end, we prove that $mag(piv_{BC}|\omega)$ is always the *restricted* magnitude $mag(piv_{BC}^{*}|\omega)$.

Mutatis mutandis, the derivation of the critical values α_{BC}^{**} and λ_{BC}^{**} is identical to that of α_{AC}^{**} and λ_{AC}^{**}, which yields:

$$\begin{aligned}\alpha_{BC}^{**} &= \frac{\tau\left(B|\omega\right)}{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)} \\ \lambda_{BC}^{**} &= \frac{\sqrt{\tau\left(C\right)\left[\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)\right]}}{2\sqrt{\tau\left(C\right)\left[\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)\right]}+\tau\left(A|\omega\right)},\end{aligned}$$

and the magnitude $mag(piv_{BC}|\omega)$ would be unrestricted iff, in these points:

$$\lambda_{BC}^{**}>1/(2+\alpha_{BC}^{**}) \tag{23}$$

To show that the latter inequality can never hold, we take the equivalent to (22) and show that:

$$\sqrt{\frac{\tau\left(C\right)}{\tau\left(B|\omega\right)+\tau\left(AB|\omega\right)}}<\frac{\tau\left(A|\omega\right)}{\tau\left(B|\omega\right)},$$

in which both fractions are larger than one. The latter implies that: $\frac{\tau(A|\omega)}{\tau(B|\omega)}\geq\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}\geq\sqrt{\frac{\tau(A|\omega)+\tau(AB|\omega)}{\tau(B|\omega)+\tau(AB|\omega)}}$, and by (20), the last member of this inequality is always larger than $\sqrt{\frac{\tau(C)}{\tau(B|\omega)+\tau(AB|\omega)}}$, which proves that $mag(piv_{BC}|\omega)$ is always restricted. This implies that $mag(piv_{BC}|\omega)$ is equal to $mag(piv_{AB}|\omega)$ if the both are restricted, and strictly smaller than $mag(piv_{AB}|\omega)$ if the latter is not restricted. ■

Proof of Proposition 3

Proof. The first step of the proof is to show that, if C is expected to rank first or second, we automatically have $G(AB|t) > \min\left[G(A|t), G(B|t)\right]$ for both types $t \in \{t_A, t_B\}$. Then, we proceed by contradiction and show that, in equilibrium, we must have $\min\left\{G(A|t_A) - G(AB|t_A), G(B|t_B) - G(AB|t_B)\right\} = 0$, which excludes the possibility C be expected to rank first or second in equilibrium.

From $(5) - (7)$, we have:

$$\begin{aligned} G(A|t) &\geq G\left(AB|t\right) \Longleftrightarrow \frac{q\left(a|t\right)}{q\left(b|t\right)} \geq M_1 \equiv \tfrac{\Pr(piv_{AB}|b)+2\Pr(piv_{BC}|b)}{\Pr(piv_{AB}|a)-\Pr(piv_{BC}|a)} \text{ if } \Pr\left(piv_{AB}|a\right) > \Pr\left(piv_{BC}|a\right) && (24) \\ G(B|t) &\geq G\left(AB|t\right) \Longleftrightarrow \frac{q\left(a|t\right)}{q\left(b|t\right)} \leq M_2 \equiv \tfrac{\Pr(piv_{BA}|b)-\Pr(piv_{AC}|b)}{\Pr(piv_{BA}|a)+2\Pr(piv_{AC}|a)} \text{ if } \Pr\left(piv_{BA}|b\right) > \Pr\left(piv_{AC}|b\right), && (25) \end{aligned}$$

and the reverse inequalities if $\Pr\left(piv_{AB}|a\right) < \Pr\left(piv_{BC}|a\right)$ and/or $\Pr\left(piv_{BA}|b\right) < \Pr\left(piv_{AC}|b\right)$.

By Lemma 3, if C is expected to rank first or second in state a, we have either $M_1 < 0$ or $M_2 = 0$. Similarly, if C is expected to rank first or second in state b, we have either $M_1 = \infty$ or $M_2 < 0$. In either case, at least one of the two inequalities (24) or (25) is violated for both types t_A and t_B.

Now, we proceed to show that either inequality $G(A|t_A) - G(AB|t_A) < 0$ or $G(B|t_B) - G(AB|t_B) < 0$ leads to a contradiction (points i to v). Finally, we show that $G(A|t_A) - G(AB|t_A) > 0$ and $G(B|t_B) - G(AB|t_B) > 0$ cannot either hold together (point vi).

(i) $\underline{G(A|t_A) - G(AB|t_A) < 0 \text{ and } G(B|t_B) - G(AB|t_B) < 0}$:

If both these inequalities hold, types t_A and t_B's best response is $\sigma(AB|t_A) = 1 = \sigma(AB|t_B)$. This implies that:

$$mag(piv_{AB}|a) = 0 = mag(piv_{AB}|b),$$

and $\tau\left(A|\omega\right) + \tau\left(AB|\omega\right) = \tau\left(B|\omega\right) + \tau\left(AB|\omega\right) > \tau\left(C|\omega\right)$ in both states of nature. As a consequence, $mag(piv_{AC}|\omega) = mag(piv_{BC}|\omega) < 0$ in both states of nature, which implies that $\lim_{n\to\infty} M_1 = \lim_{n\to\infty} M_2 = 1$. Since $q\left(a|t_A\right)/q\left(b|t_A\right) > 1$, (24) holds for types t_A, and since $q\left(a|t_B\right)/q\left(b|t_B\right) < 1$, (25) holds for types t_B: a contradiction.

(ii) $\underline{G(A|t_A) - G(AB|t_A) < 0 \text{ and } G(B|t_B) - G(AB|t_B) = 0}$:

Under the former inequality, types t_A's best response is $\sigma(AB|t_A) = 1$, whereas $\sigma(AB|t_B) \in [0, 1]$. This implies:

$$0 \leq [\tau(B|a) + \tau(AB|a)] - [\tau(A|a) + \tau(AB|a)] \leq [\tau(B|b) + \tau(AB|b)] - [\tau(A|b) + \tau(AB|b)],$$

and hence $mag(piv_{AB}|a) > mag(piv_{AB}|b)$. Two subcases must be considered.

Either $mag(piv_{AB}|a) > mag\left(piv_{BC}|a\right)$, which implies that $M_1 \underset{n\to\infty}{\to} 0$, and hence that (24) holds. This directly contradicts $G(A|t_A) - G(AB|t_A) < 0$. For the complementary subcase $mag(piv_{AB}|a) < mag\left(piv_{BC}|a\right)$, we show that $G(B|t_B) - G(AB|t_B) = 0$ cannot hold: following the same steps as in the proof of Lemma 3, one can check that $\Pr\left(piv_{AC}|b\right)$ is always the restricted pivot probability with a magnitude below that of $\Pr\left(piv_{BA}|b\right)$. Hence, $M_2 \geq 0$. Next, $mag(piv_{AB}|a) > mag(piv_{AB}|b)$ implies that $M_2 \to 0$, and hence that $G(B|t_B) - G(AB|t_B) < 0$, which leads us to case i above, and hence to a contradiction.

(iii) $\underline{G(A|t_A) - G(AB|t_A) = 0 \text{ and } G(B|t_B) - G(AB|t_B) < 0}$:

Similar to point (ii).

(iv) $\underline{G(A|t_A) - G(AB|t_A) < 0 \text{ and } G(B|t_B) - G(AB|t_B) > 0}$:

The voters' best responses are then $\sigma(AB|t_A) = 1 = \sigma(B|t_B)$, which implies

$$0 \leq [\tau(B|a) + \tau(AB|a)] - [\tau(A|a) + \tau(AB|a)] \leq [\tau(B|b) + \tau(AB|b)] - [\tau(A|b) + \tau(AB|b)],$$

which is a subcase of point (ii) above.

(v) $\underline{G(A|t_A) - G(AB|t_A) > 0 \text{ and } G(B|t_B) - G(AB|t_B) < 0}$:

Similar to point (iv).

Finally, we show that we cannot either have:

(vi) $\underline{G(A|t_A) - G(AB|t_A) > 0 \text{ and } G(B|t_B) - G(AB|t_B) > 0}$:

The best responses are then: $\sigma(A|t_A) = 1 = \sigma(B|t_B)$. Since $|r(t_A|a) - r(t_B|a)| > |r(t_B|b) - r(t_A|b)|$, we have that $mag(piv_{AB}|b) > mag(piv_{AB}|a)$ and hence that $M_1 \to \infty$, which implies that (24) is violated, a contradiction. ■

Lemma 5 and Proof

Lemma 5 *In equilibrium,* $\tau(A|b)+ \tau(AB|b) < \tau(B|a)+ \tau(AB|a)$.

Proof. By contradiction: by Lemma **??** we know that in equilibrium, $\sigma\left(A|t_A\right) = \sigma\left(B|t_B\right)\rho$ with $\rho = \frac{\left(\sqrt{r(t_B|a)}+\sqrt{r(t_B|b)}\right)^2}{\left(\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}\right)^2}$. Then the expected vote share of A in state b and of B in state a can be written:

$$\begin{aligned} \tau(A|b) + \tau(AB|b) &= r(t_A|b) + [1 - \sigma\left(B|t_B\right)]r(t_B|b) \\ \tau(B|a) + \tau(AB|a) &= r(t_B|a) + [1 - \sigma\left(B|t_B\right)\rho]r(t_A|a) \end{aligned}$$

and $\tau(A|b)+ \tau(AB|b) \geq \tau(B|a)+ \tau(AB|a)$ is equivalent to:

$$r(t_A|b) + [1 - \sigma\left(B|t_B\right)]r(t_B|b) \geq r(t_B|a) + [1 - \sigma\left(B|t_B\right)\rho]r(t_A|a)$$

Since $r(t_A|b) + r(t_B|b) = \alpha = r(t_B|a) + r(t_A|a)$, this is equivalent to

$$\begin{aligned} \alpha - \sigma\left(B|t_B\right) r(t_B|b) &\geq \alpha - \sigma\left(B|t_B\right) x r(t_A|a) \\ \frac{\sqrt{r\left(t_B|a\right)} + \sqrt{r\left(t_B|b\right)}}{\sqrt{r\left(t_A|a\right)} + \sqrt{r\left(t_A|b\right)}} &\geq \frac{\sqrt{r\left(t_B|b\right)}}{\sqrt{r\left(t_A|a\right)}}. \end{aligned}$$

Taking squares and simplifying, this boils down to:

$$\begin{aligned} [\alpha - r(t_A|a)]r(t_A|a) &\geq [\alpha - r\left(t_B|b\right)]r\left(t_B|b\right) \\ [r(t_A|a) - r\left(t_B|b\right)]\,\alpha &\geq r(t_A|a)^2 - r\left(t_B|b\right)^2 \\ \alpha &\geq r(t_A|a) + r\left(t_B|b\right) \end{aligned}$$

a contradiction. ■

Proof of Lemma 4

Proof. We begin by proving that, in equilibrium:

$$mag\left(piv_{AB}|a\right) = mag\left(piv_{AB}|b\right) \geq \max\{mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right), mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\}, \tag{26}$$

and then complete the proof of the proposition by showing that $\sigma\left(A|t_A\right)/\sigma\left(B|t_B\right)$ takes the value specified in the proposition.

Proposition 3 proved that, if an equilibrium exists, we must have $\min\left\{G(A|t_A) - G(AB|t_A), G(B|t_B) - G(AB|t_B)\right\} = 0$.

Knowing that

$$\begin{aligned} G\left(A|t_A\right) - G\left(AB|t_A\right) &= q\left(a|t_A\right)\left[\Pr\left(piv_{AB}|a\right) - \Pr\left(piv_{BC}|a\right)\right] - q\left(b|t_A\right)\left[\Pr\left(piv_{AB}|b\right) + 2\Pr\left(piv_{BC}|b\right)\right], \\ G\left(B|t_B\right) - G\left(AB|t_B\right) &= q\left(b|t_B\right)\left[\Pr\left(piv_{BA}|b\right) - \Pr\left(piv_{AC}|b\right)\right] - q\left(a|t_B\right)\left[\Pr\left(piv_{BA}|a\right) + 2\Pr\left(piv_{AC}|a\right)\right]. \end{aligned}$$

Proposition 3 requires:

$$\begin{aligned} mag\left(piv_{AB}|a\right) &\geq \max\left\{mag\left(piv_{BC}|a\right), mag\left(piv_{BC}|b\right), mag\left(piv_{AB}|b\right)\right\}, \text{ and} \\ mag\left(piv_{AB}|b\right) &\geq \max\left\{mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right), mag\left(piv_{AB}|a\right)\right\}, \end{aligned}$$

with at least one equality. Together, these imply (26).

In turn, $mag\left(piv_{AB}|a\right) = mag\left(piv_{AB}|b\right)$ imposes:

$$\begin{aligned} \left(\sqrt{\tau\left(A|a\right)} - \sqrt{\tau\left(B|a\right)}\right)^2 &= \left(\sqrt{\tau\left(A|b\right)} - \sqrt{\tau\left(B|b\right)}\right)^2 \\ \left|\sqrt{r\left(t_A|a\right)\cdot\sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|a\right)\cdot\sigma\left(B|t_B\right)}\right| &= \left|\sqrt{r\left(t_A|b\right)\cdot\sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|b\right)\cdot\sigma\left(B|t_B\right)}\right|, \end{aligned}$$

which may have up to two solutions. We show that only one of them is possible. First, we may have:

$$\sqrt{r\left(t_A|a\right)\cdot\sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|a\right)\cdot\sigma\left(B|t_B\right)} = \sqrt{r\left(t_B|b\right)\cdot\sigma\left(B|t_B\right)} - \sqrt{r\left(t_A|b\right)\cdot\sigma\left(A|t_A\right)}.$$

This holds if :

$$\sigma\left(A|t_A\right) = \sigma\left(B|t_B\right)\frac{\left(\sqrt{r\left(t_B|a\right)} + \sqrt{r\left(t_B|b\right)}\right)^2}{\left(\sqrt{r\left(t_A|a\right)} + \sqrt{r\left(t_A|b\right)}\right)^2}. \tag{27}$$

The other possibility is that one of the two members has a different sign. Then, the equality becomes:

$$\sqrt{r\left(t_A|a\right)\cdot\sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|a\right)\cdot\sigma\left(B|t_B\right)} = \sqrt{r\left(t_A|b\right)\cdot\sigma\left(A|t_A\right)} - \sqrt{r\left(t_B|b\right)\cdot\sigma\left(B|t_B\right)}.$$

Rearranging terms yields:

$$\underbrace{\left(\sqrt{r\left(t_A|a\right)} - \sqrt{r\left(t_A|b\right)}\right)}_{>0}\sqrt{\sigma\left(A|t_A\right)} = \underbrace{\left(\sqrt{r\left(t_B|a\right)} - \sqrt{r\left(t_B|b\right)}\right)}_{<0}\sqrt{\sigma\left(B|t_B\right)},$$

which is clearly impossible. Therefore (27) is the only solution to (26), which proves unicity. ■

8.1 Proof of Proposition 4

Proof. First of all, note that, for any $\sigma(B|t_B) \in (0,1]$, we have $\sigma(A|t_A) \in (0,1)$. The latter can only be an equilibrium if $G(A|t_A) = G(AB|t_A)$, i.e. if:

$$q(a|t_A) \cdot [\Pr(piv_{AB}|a) - \Pr(piv_{BC}|a)] = q(b|t_A) \cdot [\Pr(piv_{AB}|b) + 2\Pr(piv_{BC}|b)]. \tag{28}$$

Second, denote $\rho \equiv \frac{\left(\sqrt{r(t_B|a)}+\sqrt{r(t_B|b)}\right)^2}{\left(\sqrt{r(t_A|a)}+\sqrt{r(t_A|b)}\right)^2}$. For $\sigma(A|t_A) = \rho\,\sigma(B|t_B)$, we have:

$$\begin{aligned} \tau(A|\omega) &= r(t_A|\omega) \cdot \sigma(A|t_A) = r(t_A|\omega) \cdot \rho \cdot \sigma(B|t_B) \\ \tau(B|\omega) &= r(t_B|\omega) \cdot \sigma(B|t_B), \end{aligned}$$

and substituting for these values in the unrestricted magnitude of the pivotability between A and B derived in Lemma 1, it is immediate to check that they imply:

$$mag(piv_{AB}|a) = mag(piv_{AB}|b),$$

and:

$$\begin{cases} \tau(A|a) + \tau(AB|a) > \tau(B|a) + \tau(AB|a), \\ \tau(B|b) + \tau(AB|b) > \tau(A|b) + \tau(AB|b). \end{cases} \tag{29}$$

Third, for $\sigma(B|t_B)$ sufficiently close to zero, we have $mag(piv_{AB}|a) = mag(piv_{AB}|b) \to 0$ and hence $mag(piv_{AB}|\omega) > \max[mag(piv_{BC}|a), mag(piv_{BC}|b)]$. In that case, (28) boils down to:

$$q(a|t_A) \cdot \Pr(piv_{AB}|a) = q(b|t_A) \cdot \Pr(piv_{AB}|b). \tag{30}$$

From this condition, it is immediate to see that, if $\sigma(A|t_A) > \rho\,\sigma(B|t_B)$, the left hand side of this equation becomes smaller than the right-hand side, which implies that an AB-approval would be more valuable than a single-A approval, and conversely if $\sigma(A|t_A)$ is smaller than that value. Therefore, $\sigma(A|t_A) = \rho\,\sigma(B|t_B)$ is a best response for types t_A, when $\sigma(B|t_B)$ sufficiently small to have $mag(piv_{BC}|\omega) < mag(piv_{AB}|a)$, and it is "stable".

Next, we turn to types t_B. $\sigma(B|t_B) \in (0,1)$ can be an equilibrium if there exists a value between zero and one that yields $G(B|t_B) = G(AB|t_B)$, i.e.:

$$q(a|t_B) \cdot [\Pr(piv_{BA}|a) + 2\Pr(piv_{AC}|a)] = q(b|t_B) \cdot [\Pr(piv_{BA}|b) - \Pr(piv_{AC}|b)]. \tag{31}$$

For σ_B close to zero, this condition would again boil down to:

$$q(a|t_B) \cdot \Pr(piv_{BA}|a) = q(b|t_B) \cdot \Pr(piv_{BA}|b), \tag{32}$$

but this condition cannot hold together with (30). Indeed, by Myerson's offset theorem, for sufficiently large populations, we have:

$$\Pr(piv_{BA}|\omega) = \Pr(piv_{AB}|\omega)\ \sqrt{\tau(A|\omega)/\tau(B|\omega)}.$$

Substituting for these values in condition (32), and for the value of $q(\omega|t_B)$, we see that the condition could only hold if:

$$(1 >)\frac{r(t_B|a)}{r(t_B|b)} = \underbrace{\frac{\Pr(piv_{AB}|b)\sqrt{r(t_A|b)}}{\Pr(piv_{AB}|a)\sqrt{r(t_A|a)}} = \sqrt{\frac{r(t_A|a)}{r(t_A|b)}}}_{\text{By condition (30)}}(> 1),$$

which is impossible. Instead, this shows that $G(B|t_B) > G(AB|t_B)$ whenever $\sigma\left(B|t_B\right)$ is sufficiently small to have $mag\left(piv_{AC}|\omega\right) < mag\left(piv_{AB}|a\right)$. Conversely, by Proposition 3, the magnitude of $\Pr\left(piv_{AC}|\omega\right)$ cannot either be larger than that of $\Pr\left(piv_{BA}|\omega'\right)$, for any ω and $\omega' \in \{a,b\}^2$. Indeed, this would imply $G(B|t_B) < G(AB|t_B)$, which cannot be an equilibrium, by the same proposition.

Two cases may thus occur. Either there exists a value $\sigma^* \in (0,1)$ such that, for $\sigma\left(B|t_B\right) = \sigma^*$ and $\sigma\left(A|t_A\right) = \rho\,\sigma^*$:

$$mag\left(piv_{BA}|a\right) = mag\left(piv_{BA}|b\right) = \max\left[mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\right].$$

In that case, σ^* is an equilibrium for types t_B. Moreover, by Lemma 5 in the appendix, we have $mag\left(piv_{BC}|b\right) < mag\left(piv_{AC}|a\right)$. It is also clear that $mag\left(piv_{BC}|a\right)$ is yet smaller, because B and C are the bottom-two contenders in state a. This implies that condition (30) is still valid for types t_A, and hence that $\sigma\left(A|t_A\right) = \rho\,\sigma^*$ is an equilibrium as well.

If such a value σ^* does not exist, then $\sigma\left(B|t_B\right) = 1$ is the equilibrium strategy of types t_B: for any $\sigma\left(B|t_B\right) \in [0,1]$, we have:

$$mag\left(piv_{BA}|a\right) = mag\left(piv_{BA}|b\right) > \max\left[mag\left(piv_{AC}|a\right), mag\left(piv_{AC}|b\right)\right],$$

and hence $G(B|t_B) > G(AB|t_B)$. It is also an equilibrium of the game since (30) is still applies in $\sigma\left(A|t_A\right) = \rho$.

Finally, by Lemma 3, this excludes that C is either first or second in either state of nature. Therefore, the equilibrium is Condorcet-like:

$$\begin{cases} \tau(A|a) + \tau(AB|a) > \max\left[\tau(B|a) + \tau(AB|a), \tau\left(C\right)\right], \\ \tau(B|b) + \tau(AB|b) > \max\left[\tau(A|b) + \tau(AB|b), \tau\left(C\right)\right]. \end{cases} \tag{33}$$

■