Shouting to be heard in advertising

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Abstract

Advertising competes for consumer attention, but attention is a scarce resource. Endogenizing the number of messages per sender means more profitable senders send more messages. There may be multiple equilibria: when there are more messages in aggregate, there is more "shouting to be heard" among individual senders trying to break through the advertising clutter, which creates a "lottery ticket" dimension for profit dissipation. Equilibria can be Pareto ranked as long as receiver surplus is not more elastic than sender surplus, with all agents preferring less shouting. Increasing the cost of sending messages can make all senders and the receiver better off.

JEL CLASSIFICATION: D11, D60, L13.

KEYWORDS: information overload, congestion, advertising, common property resource, overfishing, lottery, junk mail, e-mail, telemarketing, multiple equilibria, Pareto improvement.

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1 Introduction: more than one message?

Consumers are bombarded with advertisements enticing them to buy wares. Advertising messages are embedded in television programs in the evening, in radio programs during the morning commute, in the newspaper read in the coffee break, and as pop-up ads while net-surfing. When driving home, they are present in billboards and neon signs. Consumers can scarcely process all the incoming information, and typically screen out a lot of the messages projected at them. Advertisers need their messages to break through the advertising clutter of others' messages in order to attract attention and eventually get to a sale. In such a world, advertisers are more likely to make a connection if they send several messages. This paper is about the number of messages sent and the welfare properties of the market solution.

> Well, it's a non-stop blitz of advertising messages. Everywhere we turn we're saturated with advertising messages trying to get our attention. We've gone from being exposed to about 500 ads a day back in the 1970's to as many as 5,000 a day today. We have to screen it out because we simply can't absorb that much information. We can't process that much data and so no surprise, consumers are reacting negatively to the kind of marketing blitz; the kind of super saturation of advertising that they're exposed to on a daily basis. [The conundrum for modern day marketers is how to cut through the clutter.] All of this marketing saturation that's going on is creating this kind of arms race between marketers where they have to up the ante the next time out because their competitors have upped the ante the last time they were out. And the only way you can win is to have more saturation.
>
> President of the Marketing Firm Yankelovich, Jay Walker-Smith.[1]

The model describes a continuum of senders of messages facing a potential consumer whose attention is restricted to only process a limited number of messages. Sending more messages is more costly for a sender, but increases the probability of breaking through. With a fixed consumer attention span, sending messages is like a lottery where there are multiple winners. Participants typically value a winning connection

[1] Excerpted from Cutting Through Advertising Clutter, CBS News
http://www.cbsnews.com/stories/2006/09/17/sunday/main2015684.shtml

differently because they enjoy different profits. Buying more tickets (sending more messages) improves the odds of breakthrough, but reduces the chances for other senders. Since we assume that only one message has to connect to make a sale, and that the benefit from a connection is independent of which other connections are made, winning the lottery twice (or more) confers no further benefit beyond winning once, because the consumer has already got the message. That is, two successful ads are not better than one.

The model of senders trying to contact a receiver builds on the framework developed by Van Zandt (2004) and Anderson and de Palma (2007). These papers assume that each sender can send at most one message to the receiver. Here we analyze market performance when senders can send many messages. Allowing for multiple messages brings up the possibility of overcrowding in both depth and width. The width decision gives a further dimension of the common property problem.

The literature on informative advertising, starting with Butters (1977) assumes that senders within the same industry send messages announcing the availability and price of a good. Messages are sent randomly to consumers, so each consumer will get a different profile of messages. In this context, each sender sends a single message, which is viewed as a zero-profit proposition under the assumption of free-entry of senders. In Grossman and Shapiro (1984), senders form an oligopoly (again within the same industry), and each firm sends multiple messages, again randomly distributed over receivers, in order to reach a desired customer base. All messages are assumed to be read by the recipient, and all senders are symmetric. By contrast, our approach emphasizes the advertising communication technology and looks at heterogenous senders from different industries (as opposed to the impact of advertising within a particular industry) interacting through a bottleneck of limited individual attention span. In this context, senders send multiple messages even though they know to whom they are sending their messages. Extending our model to allow for uncertainty about whether the recipient got the message (which is the reason for sending multiple messages in Grossman and Shapiro, 1984) ought not greatly alter our conclusions, and we suppress this to focus on the limited attention span motive for multiple messages.

The paper proceeds by first analyzing the problem of a single sender choosing an integer number of messages as a function of its profitability, taking as given the mass of messages sent by the others. This procedure yields a step function for the number of messages sent: more profitable senders send more messages. We solve the model by starting with the marginal seller just indifferent between sending one message or none at all. This sender determines a congestion level, which then determines the critical marginal senders for all other step sizes. Integrating up yields the aggregate number of messages sent, and hence the examination level consistent with the earlier congestion level and the seed value of the first marginal sender. The examination value is a continuous function whose range is the positive real line. We are then able to prove a first marginal sender, and hence an equilibrium, exists for any value of examination cost. There may indeed be multiple equilibria, with higher conditional profitability values of the marginal sender corresponding to more messages, which underlies the property of "shouting to be heard."

For the normative analysis, we take a continuous approximation to the step function for messages and first argue that a proportionately lower message volume improves welfare by reducing clutter and duplication of message reception. We then show that each equilibrium message density (as a function of sender type) has a unique intersection with each other one even when they are normalized (by proportional reductions) to comprise the same aggregate message volume. This allows us to then show that lower volumes are preferable so long as receiver surplus is no less elastic than profitability with respect to sender type.

In this context, increasing the cost of sending each message might be expected to help (for those receivers who do not examine all messages) in two dimensions. First, one might expect there to be fewer sender types (the low end drops off) and less competition (in terms of messages sent) at any given level. Instead, however, we show it is possible that raising the cost of sending messages causes more senders to send, and fewer messages in total. This cost increase actually makes everyone better off, even if the extra cost is not rebated to senders. Even highly profitable sender types that send a lot of messages see profits rise through an increase in the probability they are examined. The high types now are more "prominent" in the sense that their messages get through more clearly (less clutter from other types), and efficiency is improved.

The next section sets out the model and provides the first distinctive contribution of the analysis, which is the equilibrium determination of the integer problem of how many messages are sent per sender. Section 3 then characterizes the multiplicity of equilibria. Section 4 provides the second distinctive analysis, which compares the welfare properties of the equilibria. Section 5 addresses the effects of changing the cost of transmission, applying the same analytical tools. Section 6 offers some conclusions.

2 Breaking through Advertising Clutter

2.1 Breakthrough probability

There is a single receiver, who is willing to examine a fixed number of messages, ϕ; all are examined if no more than ϕ messages are received. If a message is examined, a further message from the same sender has no impact.[2] The assumption of a constant ϕ is broadly consistent with some empirical evidence from the marketing literature: Brown and Rothschild (1993) attribute this finding to Webb and Ray (1979).[3]

There is a continuum of senders, with total mass M, and they are ranked by the profitability conditional on getting a message through to the receiver. This conditional profitability is the product of the probability the receiver is interested (will buy the product) and the profit conditional on being interested, and there is no contagion across senders. A sender's rank (or type) is denoted by $\theta \in [0, 1]$, with associated expected profit $\pi(\theta) > 0$. We assume this expected profit is continuously differentiable and strictly increasing with $\pi(0) < \gamma < \pi(1)$, where γ is the cost to sending a message. These bounds on $\pi(.)$ ensure the market is neither unserved nor completely covered. Let n denote the total number of messages sent by all senders, with $\phi < n$. Otherwise (if $\phi \geq n$) there is no congestion and only one message will be sent by each sender, a case that will be dealt with when it arises below. We thus assume that senders are independent in the sense that the receiver's purchase decisions are independent of whether she has purchased other products, and which other products. This simplifying assumption allows us to concentrate on the congestion of messages in the receiver's attention span, without worrying about direct "business stealing" across messages.

Suppose that a sender transmits ℓ messages to the receiver, and we will later determine ℓ endogenously. The probability that one of its messages is examined by the receiver is given as follows.

First, since message provenance is undetermined ex ante, the probability that a message is examined is $\phi/n < 1$. Then $(1 - \phi/n)$ is the probability that the first of the ℓ messages sent by the sender is *not* examined by the individual. Likewise, $(1 - \phi/(n-1))$ is the probability that the second of its messages sent is not examined by the individual, given that one of the n messages has been examined (search without

[2] The marketing maxim contends that several hits per consumer (estimates vary from 3 to 16) are needed to get a reasonable chance of a sale. This maxim is consistent with the idea that a message needs to break through the clutter into the individual's attention span, as per our model. It is also consistent with the idea that messages build on each other to reach a critical mass to be recognized, or indeed that more messages build upon each other to persuade more gradually.

[3] Brown and Rothschild (1993) then go on to reconsider it in the light of their own further experiments, and suggest there may be a less severe congestion effect at high levels of clutter. We emphasize the attention span bottleneck by assuming ϕ is constant, but we conjecture the main results still hold true if ϕ/n were decreasing in the number of messages, n. Note that ϕ/n is measured as the ad recall rate in the marketing studies.

replacement). Therefore, the probability that none of the ℓ messages are examined is the product of ℓ terms representing sequentially missing each message.

$$\tilde{\mathbb{P}}(n,\ell,\phi) = \left(1-\frac{\phi}{n}\right)...\left(1-\frac{\phi}{n-(\ell-1)}\right) \tag{1}$$

The probability that the receiver examines at least one of the messages sent by the sender is $1-\tilde{\mathbb{P}}(n,\ell,\phi)$, and, in this event the message content is absorbed regardless of how many other messages are received from the same sender. If the number of messages sent by a sender is small (i.e., ℓ is small relative to n, as will be true in the monopolistic competition set-up used below), then the probability that none of the ℓ messages sent by a sender is examined is approximately $[1-\phi/n]^{\ell}$.[4] This is the exact formula when messages are examined with replacement, which is a common assumption in the search literature.

The probability, $\mathbb{P}(n,\ell,\phi)$, that at least one of the sender's messages is examined by the receiver is then

$$\mathbb{P}(n,\ell,\phi) = 1-[1-\phi/n]^{\ell}. \tag{2}$$

This formula is exact when there is a continuum of senders, since then an individual sender's choice of l has infinitesimal impact on aggregate measures.

2.2 How many messages per sender?

Senders consider n as fixed when choosing how many messages they should send. This is akin to a monopolistic competition assumption: each sender is a small contributor to the total number of messages sent.

Then sender θ's problem is

$$\max_{\ell}\left\{\pi(\theta)\left[1-\left(1-\frac{\phi}{n}\right)^{\ell}\right]-\gamma\ell\right\}, \qquad \ell\geq 0, \tag{3}$$

where γ is the cost of sending a message.

We take explicit account of the constraint that the number of messages must be an integer. Sender θ prefers to send ℓ rather than $\ell-1$ messages if the extra benefit of doing so exceeds the extra cost, i.e., if:

$$\pi(\theta)\left[\left(1-\frac{\phi}{n}\right)^{\ell-1}-\left(1-\frac{\phi}{n}\right)^{\ell}\right]>\gamma, \qquad \ell\geq 1,$$

This may be expressed as

$$\pi(\theta)\left(1-\frac{\phi}{n}\right)^{\ell-1}\left(\frac{\phi}{n}\right)>\gamma, \qquad \ell\geq 1. \tag{4}$$

[4] This is similar to Butters' (1977) classic model of letterbox advertising.

A sender of type θ will transmit at least one message if $\pi(\theta)\left(\frac{\phi}{n}\right) > \gamma$ (as seen below, the value of θ for which this holds with equality is the marginal sender type, θ_1, and types $\theta < \theta_1$ do not transmit).

The interpretation of the L.H.S. of (4) is that $\left(1-\frac{\phi}{n}\right)^{\ell-1}$ represents the probability that sender θ's first $(\ell-1)$ messages are not examined, and $\left(\frac{\phi}{n}\right)$, the inverse level of congestion, is the probability that the last message, the marginal ℓ-th one, *is* examined. Note that the L.H.S. is decreasing in ℓ, indicating that the profit function is concave, and the L.H.S. is increasing in θ. This means that the higher θ senders send out more ads. Since there are decreasing returns to messages sent and $\pi(.)$ is increasing, there are rents to higher θ senders, which make higher profits. (This latter property can be seen readily by applying the envelope theorem to (3)).

For any ϕ and n, sender θ will prefer sending j to any smaller number of messages as long as

$$\pi(\theta) > \frac{n}{\phi}\frac{\gamma}{\left(1-\frac{\phi}{n}\right)^{j-1}}. \tag{5}$$

Hence, higher-value senders (those with higher types) transmit no fewer messages than lower value senders. The sender θ_j which is indifferent between sending $j-1$ and j messages has type such that

$$\pi(\theta_j) = \frac{n}{\phi}\frac{\gamma}{\left(1-\frac{\phi}{n}\right)^{j-1}}. \tag{6}$$

The R.H.S. of this expression is increasing in j while the L.H.S. is increasing in θ_j. As a consequence, the number of messages sent is step-function with step size one. The marginal sender type, θ_1, is defined from (6) as the type which is indifferent between sending one or no messages, so

$$\pi(\theta_1) = \frac{n}{\phi}\gamma. \tag{7}$$

Substituting (7) into (6) yields an implicit expression for θ_j in terms solely of θ_1:

$$\pi(\theta_j) = \frac{\pi(\theta_1)}{\left(1-\frac{\gamma}{\pi(\theta_1)}\right)^{j-1}}, \qquad j = 1, ..., k+1, \tag{8}$$

where k is such that $\pi(\theta_k) \leq \pi(1) < \pi(\theta_{k+1})$, and is the maximum number of messages sent per sender.

This procedure will conveniently allow us to define the full solution. To recapitulate:

Lemma 1 *The number of messages sent per sender is an increasing step-function with step-size one, with the first step starting at* θ_1 *satisfying (7).*

More profitable sender types (θ larger than θ_1) have a greater incentive to duplicate, and send a greater volume.

In the sequel we will use a linear profit function example, $\pi(\theta) = \bar{\pi}\theta$, to illustrate various properties. Figure 1 displays the critical θ's as a function of θ_1 using (8) where $\gamma/\bar{\pi} = 1/20$). The black line (45 degree line) is θ_1, the next line up (red) is the locus for $j = 1$, followed by $j = 2$ (dark green), etc. For example, any candidate solution with $\theta_1 = 0.4$, entails 5 or more messages being sent for θ around 0.8, and the maximum number of messages reaching 6 for all senders whose profitability exceeds (approximately) $0.9\bar{\pi}$. The critical θ's are single-troughed functions of θ_1, which is a general property of these functions.

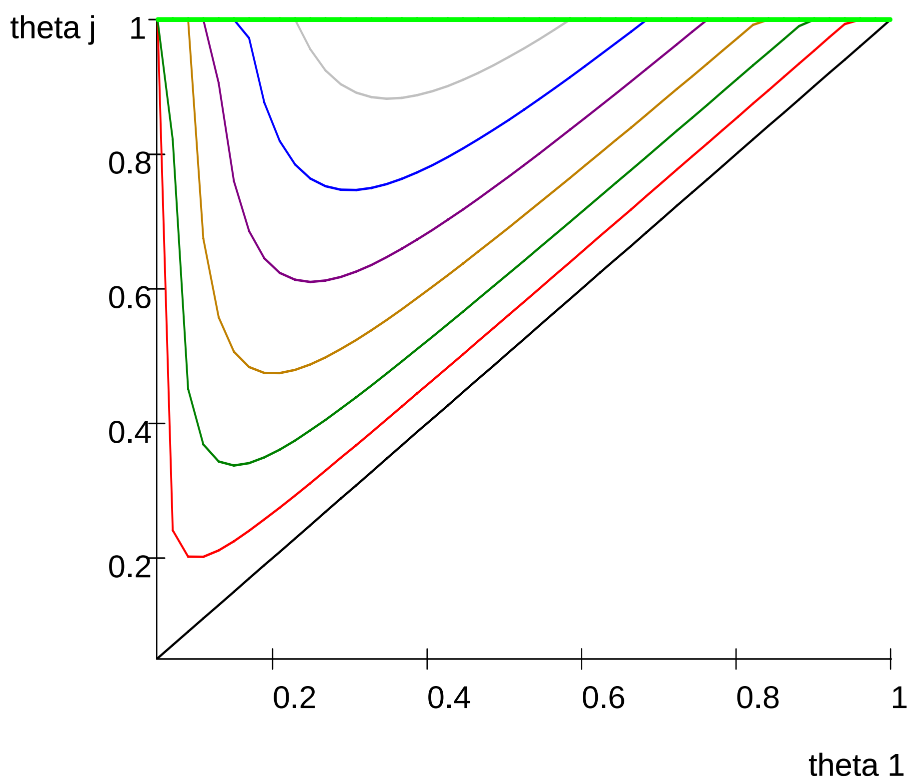


Figure 1. Critical points for steps, θ_j, $j = 1, ..., 7$. Red is lowest θ for 2 steps (i.e., θ_2 as a function of θ_1), green for 3 steps, etc.

Recall the maximum number of messages sent per sender is denoted by k and this is the number of messages sent by type $\theta = 1$. This is seen from Figure 1 to be first increasing and then decreasing in θ_1 (from 1 to 6 and then back down to 1: track the light green line at the top of the Figure). The idea is that with θ_1 small the marginal sender has a low profitability, and can only be induced to send a message if there is very little congestion, which in turn means that each other sender does not want to send many messages because there is a high chance the first one will get through. At the other extreme, if the marginal active type (indifferent between sending zero and one message) is high, then there are few higher types and they are unlikely to want to send several messages given the marginal type did not (unless the profit as a function of type becomes very elastic).[5]

We return now to the general analysis. The equilibrium solution boils down to a triple of endogenous values for n, k, and θ_1 for a given attention span ϕ and for a profitability function $\pi(\theta)$. This requires finding the set of active senders as well as the number of messages sent by each sender. There may be multiple equilibria, meaning that the set of active senders is not uniquely determined from the value of ϕ.

The solution method we use is as follows. We consider a solution characterized by the identity of the marginal active sender θ_1 with $\theta_1 \in [0, 1]$. We then compute the congestion level, n/ϕ, the number of senders, n, and the attention span, ϕ, required to support such a solution. We show that there exists a attention span which supports an equilibrium characterized by the marginal type θ_1. This attention span function, denoted $\Phi(\theta_1)$, is continuous on the domain $(\pi^{-1}(\gamma), 1)$, although it is not differentiable everywhere since it involves different regimes characterized by the maximal number of messages sent, k. The function $\Phi(\theta_1)$ allows us to back out the value or values of θ_1 that can be sustained as equilibria for any ϕ.

We first deal with two simple cases where $\theta_1 \leq \pi^{-1}(\gamma)$.

Lemma 2 *Consider an equilibrium candidate with θ_1 as the marginal sender type. Then:*

i) There can be no equilibrium with $\theta_1 < \pi^{-1}(\gamma)$;

ii) A marginal sender type $\theta_1 = \pi^{-1}(\gamma)$ can be supported as an equilibrium if and only if $\phi \geq [1 - \theta_1] M$. Such an equilibrium entails $k = 1$.

[5] With high θ_1, there must be a relatively high congestion level, which at first glance would seem to suggest a high level of messages per sender. However, recall that the congestion level is a ratio of messages sent to messages examined, so that a high congestion level can be consistent with a low volume of messages sent, and that is what is happenign here.

Proof: i) Profit of any type can be at most $\pi(\theta) - \gamma$ because this is the profit if its message was examined for sure. This profit would necessarily be negative for $\theta < \pi^{-1}(\gamma)$. Hence, there can be no equilibrium with $\theta_1 < \pi^{-1}(\gamma)$.

ii) For $\theta_1 = \pi^{-1}(\gamma)$ the marginal sender's profit is zero if and only if there is no congestion, so that this is an equilibrium if and only if $\phi \geq n = [1 - \theta_1] M$. Here all active senders transmit just a single message since they are always examined (so there is no congestion).

Q.E.D.

Now consider $\theta_1 \in (\pi^{-1}(\gamma), 1]$. In what follows we use Lemmas to describe the properties of the attention span function $\Phi(\theta_1)$ and we use Propositions to give the implications for the equilibrium relationship for given ϕ. Thus, we seek the equilibrium relation between ϕ and θ_1 (for given γ). We construct the solution iteratively in a 6-step procedure, starting with a value of $\theta_1 \in (\pi^{-1}(\gamma), 1]$. This value induces a value of ϕ which then determines a solution as defined by the triple k, n and θ_1. The six steps are:

1. Select a value of $\theta_1 \in (\pi^{-1}(\gamma), 1]$.

2. The (inverse) congestion ratio is given by (7) as: $\frac{\phi}{n} = \frac{\gamma}{\pi(\theta_1)}$

3. The sequence of θ_j is given by equations (6) for $j > 1$. Substituting using step 2 leads to (8) which gives expressions for the sequence of θ_j $(j = 2, 3, ...)$ that depend only on θ_1, namely $\pi(\theta_j) = \frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{j-1}}$. The denominator of this expression is positive for $\theta_1 \in (\pi^{-1}(\gamma), 1]$. As expected, $\theta_j < \theta_{j+1}$ by monotonicity of $\pi(.)$.

4. The maximum number of messages transmitted, k, is the unique integer that satisfies $\theta_k \leq 1 < \theta_{k+1}$. Equivalently, k satisfies $\pi(\theta_k) \leq \pi(1) < \pi(\theta_{k+1})$ or:

$$\frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{k-1}} \leq \pi(1) < \frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{k}}. \tag{9}$$

Conditions (9) determine a unique (finite) value of k as a function of θ_1.

5. The number of messages transmitted is given by adding up the number of messages on each level, $j = 1, ...k$, so

$$n(\theta_1) = M[1 - \theta_1] + M[1 - \theta_2] + ...M[1 - \theta_k] > 0, \tag{10}$$

or $n(\theta_1) = Mk - M \sum_{j=1...k} \theta_j$, so that

$$n(\theta_1) = Mk - M \sum_{j=1...k} \pi^{-1} \left(\frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{j-1}} \right), \tag{11}$$

with $M[1-\theta_1] \leq n(\theta_1) \leq Mk[1-\theta_1]$, so $n(\theta_1)$ is bounded above. Note that $n(\theta_1) = M[1-\theta_1]$ when $k = 1$.

In the examples, we use the linear profit function ($\pi(\theta) = \bar{\pi}\theta$) so:

$$n(\theta_1) = Mk - M\theta_1 \sum_{j=1...k} \left[1 - \frac{\gamma}{\bar{\pi}\theta_1}\right]^{1-j}. \tag{12}$$

6. Using the value of $n(\theta_1)$ above, the value of $\Phi(\theta_1)$ can be recovered by (7):

$$\Phi(\theta_1) = \frac{n(\theta_1)}{\pi(\theta_1)}\gamma \leq n(\theta_1), \tag{13}$$

so that $\Phi(\theta_1)$ is bounded above.

Using this algorithm to compute the solution leads to the following result, which is proved in the Appendix.

Lemma 3 *For any $\theta_1 \in (\pi^{-1}(\gamma), 1]$, there exists a unique examination value $\Phi(\theta_1)$ given by (13) that supports an equilibrium with $M[1-\theta_1]$ senders transmitting messages. The function $\Phi(\theta_1)$ is continuous and differentiable almost everywhere, and there is always congestion. The highest level of messages transmitted at such an equilibrium, $k(\theta_1)$, is determined by (9), while the total number of messages transmitted, $n(\theta_1)$, is given by (11). As $\theta_1 \downarrow \pi^{-1}(\gamma)$ or as $\theta_1 \uparrow 1$, $k(\theta_1) = 1$ and $\Phi(\theta_1) = M[1-\theta_1]$.*

The properties described in the two Lemmas above enable us to show the existence of an equilibrium and some key features of this equilibrium in the next Proposition. It may be helpful for the reader to refer to Figure 2 which illustrates the function $\Phi(\theta_1)$ (for the linear example, $\pi(\theta) = \bar{\pi}\theta$, with $\bar{\pi}/\gamma = 10$) as the upper envelope of the curves drawn for $\theta_1 \in (\pi^{-1}(\gamma), 1]$, i.e., for $\theta_1 \in (0.1, 1]$. The precise details of the construction of this Figure will be described below: it is provided here for convenience in following the argument of the Proposition. Note that the black line continues up the vertical axis (at $\theta_1 = \pi^{-1}(\gamma)$, or $\theta_1 = 0.1$).

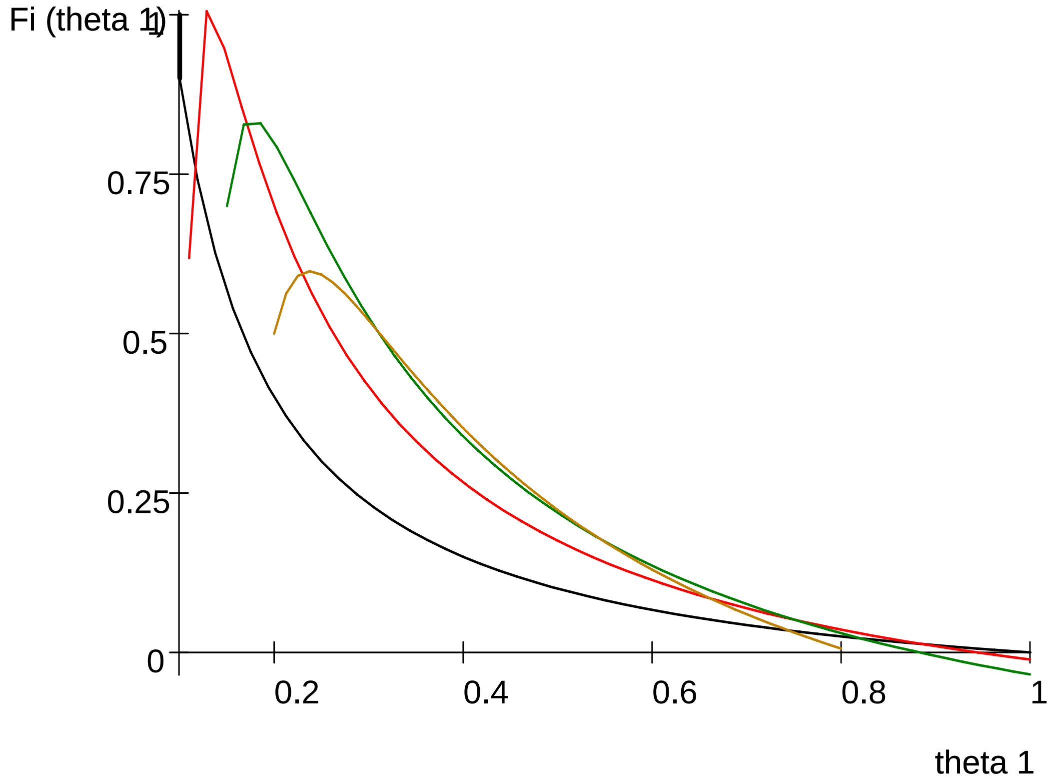


Figure 2. The function $\Phi(\theta_1)$.

Equilibrium existence is proved in the next Proposition.

Proposition 1 *For any* $\phi \geq \left[1 - \pi^{-1}(\gamma)\right] M$ *there exists an equilibrium with no congestion, where each sender transmits a single message* ($k = 1$)*. This is the unique equilibrium for* ϕ *large enough. For any* $\phi < \left[1 - \pi^{-1}(\gamma)\right] M$ *there exists an equilibrium and the equilibrium solution,* θ_1 *is continuous in* ϕ *almost everywhere. Furthermore, as* $\phi \uparrow \left[1 - \pi^{-1}(\gamma)\right] M$ *or as* $\phi \downarrow 0$*, there exists an equilibrium at which only a single message is transmitted by all active senders. The corresponding limits of the solution* θ_1 *are* $\pi^{-1}(\gamma)$ *and* 1*.*

Proof: The existence of the equilibrium type specified for $\phi \geq \left[1 - \pi^{-1}(\gamma)\right] M$ follows from Lemma 2 above: there is no congestion ($\phi \geq n$) and only one message is sent per sender. For large enough ϕ, this is the only equilibrium type because Φ is bounded on $(\pi^{-1}(\gamma), 1]$.

For $\phi < \left[1-\pi^{-1}(\gamma)\right] M$, we know that the function $\Phi(\theta_1)$ is continuous on $(\pi^{-1}(\gamma), 1]$ and tends to $\left[1-\pi^{-1}(\gamma)\right] M$ and 0 as it approaches its limits. Conversely, any $\phi \in [0, \left[1-\pi^{-1}(\gamma)\right] M)$ gives rise to a solution (or solutions) $\theta_1 \in (\pi^{-1}(\gamma), 1]$. There exists a solution which tends to $\pi^{-1}(\gamma)$ as ϕ tends to $\left[1-\pi^{-1}(\gamma)\right] M$ since the continuous function $\Phi(\theta_1)$ tends to $\left[1-\pi^{-1}(\gamma)\right] M$ as θ_1 tends to $\pi^{-1}(\gamma)$ and a similar argument establishes that the solution tends to 1 as ϕ tends to 0.

Q.E.D.

We show below that the function $\Phi(\theta_1)$ is not necessarily monotone and this feature gives rise to multiple equilibria and discontinuities in the mapping from $\phi \to \theta_1$. Before turning to the demonstration, we provide some other useful properties of the function $\Phi(\theta_1)$.

Proposition 2 *There is an odd number of equilibria for almost all ϕ.*

Proof: Equilibrium existence was proved in Proposition 1. Lemmas 2 and 3 show that Φ is continuous on $(\pi^{-1}(\gamma), 1]$. At $\theta_1 = \pi^{-1}(\gamma)$, Φ takes all values greater than or equal to $M\left[1-\pi^{-1}(\gamma)\right]$, and at $\theta_1 = 1$, $\Phi = 0$. Hence, any value of ϕ cuts the function $\Phi(\theta_1)$ an odd number of times, except when ϕ corresponds to a turning point of Φ.

Q.E.D.

Ranking the equilibria from lowest to highest θ_1 values, the "odd-numbered" ones are the ones for which the examination curve, $\Phi(\theta_1)$, is locally falling. These are readily seen to be the stable equilibria in that a perturbation in the neighborhood of an "even" equilibrium will lead away from it. This property is seen from the function $\Phi(.)$: whenever it is above ϕ, the market will support more messages, which leads n to rise (and so θ_1 to fall).

Figure 3 below illustrates multiple equilibria for a parametric example with high ϕ values (for the linear example, $\pi(\theta) = \bar{\pi}\theta$, with $\bar{\pi}/\gamma = 10$). At the level of ϕ given by the blue line, there are 5 equilibria. The first one has congestion with only one message per sender. For a higher level of ϕ (for example, $\phi = 0.95$) there are 3 equilibria. The first has one message per sender and no congestion. But the other two have two messages per sender and congestion.

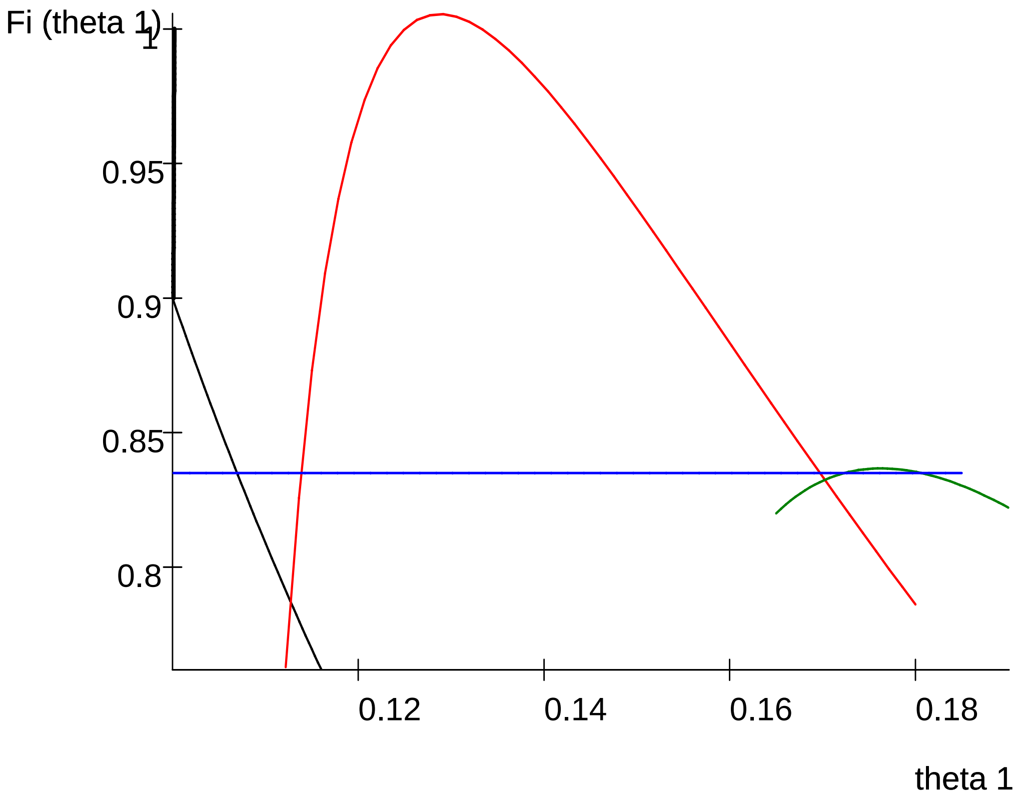


Figure 3. Illustration of 5 equilbria $k = (1{,}2{,}2{,}3{,}3)$, linear profit function.

Lemma 4 *Along the curve $\Phi(\theta_1)$ the congestion rate $n/\Phi(\theta_1)$ is strictly increasing in θ_1.*

Proof: Recall (from (13)) that $n/\Phi(\theta_1) = \pi(\theta_1)/\gamma$, which proves that $n/\Phi(\theta_1)$ is increasing in θ_1 since $\pi(\theta_1)$ is strictly increasing in θ_1.

Q.E.D.

Even though the function $\Phi(\theta_1)$ may be locally increasing, the corresponding increase in n is always large enough to raise the congestion rate, $n/\Phi(\theta_1)$, as θ_1 rises. Thus if a more profitable sender (a sender with a higher θ) is the marginal one, then congestion will be higher despite the fact that there are fewer senders! The Lemma also allows us to immediately state a comparative static property of the equilibrium.

Proposition 3 *If the attention span function, $\Phi(.)$, is locally increasing, an increase in ϕ locally raises the total number of messages sent and the congestion rate. If $\Phi(.)$, is locally decreasing, an increase in ϕ locally decreases the total number of messages sent and the congestion rate.*

The case of $\Phi(.)$ decreasing corresponds to a stable equilibrium, as noted above. Then a larger attention span eases the congestion problem as the volume of messages falls and allows in lower marginal sender types. Conversely, though, if a smaller attention span results from getting many messages, this will only exacerbate the problem as the number of messages will rise still further.

Proposition 4 ***Lemma 5*** *Along the curve $\Phi(\theta_1)$ the maximal number of messages sent per sender is given by*

$$k = \left\lceil \frac{\ln \pi(1) - \ln \pi(\theta_1)}{\ln \pi(\theta_1) - \ln(\pi(\theta_1) - \gamma)} \right\rceil$$

where $\lceil . \rceil$ denotes the ceiling function; this maximal number is either always 1, or else first increases from 1 to $k^{\max}$ with step-size one and then decreases with step-size one to 1.

Proof: First recall that the maximum number of messages, k, sent by any sender for any θ_1 is determined by Equation (9). This equation gives the solution for k as:

$$\frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{k-1}} \leq \pi(1) < \frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{k}}.$$

These bounds can be rewritten as $(k-1) \leq F(\theta_1) < k$, or

$$k = \lceil F(\theta_1) \rceil$$

with $F(\theta_1) = \frac{\ln \pi(\theta_1) - \ln \pi(1)}{\ln\left(1 - \frac{\gamma}{\pi(\theta_1)}\right)}$, where the notation $\lceil . \rceil$ denotes the ceiling function (rounding up to the next higher integer). This function is represented in Figure 4 below.

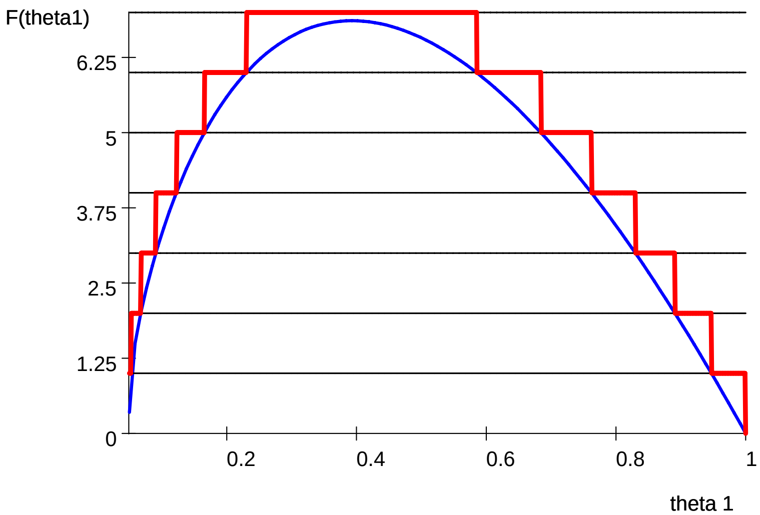


Figure 4. The function $F(\theta_1)$ (blue) and its ceiling function $\lceil F(\theta_1) \rceil$ (red) for $\bar{\pi}/\gamma = 20$.

We now prove that this function is quasi-concave in θ_1. At the end-points we have $F\left(\pi^{-1}(\gamma)\right) = 0$ and $F(1) = 0$. $F(.)$ is also continuously differentiable and positive for $\theta_1 \in (\pi^{-1}(\gamma), 1)$; therefore there is a least one turning point. We show that there is a unique solution to $F'(\theta_1) = 0$ so that $F(\theta_1)$ is quasi-concave on its support. Differentiation of $F(.)$ implies that

$$sgn\left[F'(\theta_1)\right] = sgn\left[\frac{\pi(\theta_1)}{\gamma}\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]\ln\left(1 - \frac{\gamma}{\pi(\theta_1)}\right) - \ln \pi(\theta_1) + \ln \pi(1)\right].$$

Define $x = \gamma/\pi(\theta_1)$, with $x \in (0, 1)$. Then we can rewrite

$$sgn\left[F'(\theta_1)\right] = sgn\left[\left[\frac{1}{x} - 1\right]\ln(1-x) - \ln \pi(\theta_1) + \ln \pi(1)\right].$$

Any solution to $F'(\theta_1) = 0$ solves the equation

$$\left[\frac{1}{x} - 1\right]\ln(1-x) = \ln \pi(\theta_1) - \ln \pi(1).$$

The LHS of this equation is increasing in θ_1. For the solution to $F'(\theta_1) = 0$ to be unique, it then suffices to prove that the RHS in increasing in x (since x is decreasing in θ_1). Here

$$\frac{d}{dx}\left[\left[\frac{1}{x} - 1\right]\ln(1-x)\right] = -\ln(1-x) - x > 0,$$

since $-\ln(1-x) - x$ is increasing and zero at $x = 0$. Quasi-concavity and continuity of $F(.)$ implies that the maximal number of messages sent is increasing and then decreasing in steps of size 1.

Q.E.D.

When there are multiple equilibria, we can have solutions with many active senders all sending few messages and others with fewer senders but a larger number of messages in total. The higher k is what enables the total number of messages to be higher, as higher value senders transmit more messages to get through the clutter created by their competitors.

Corollary 1 *If $k = 1$ throughout the whole range of θ_1, $\Phi(\theta_1)$ is strictly decreasing.*

Proof: If $k = 1$, from (13), $\phi(\theta_1) = M[1 - \theta_1]\gamma/\pi(\theta_1)$, which is strictly decreasing.

Q.E.D.

This Corollary implies that equilibrium is unique in this case. Notice that this also covers the case when senders are restricted to send either one or no messages.

Proposition 5 *If $\gamma \geq \frac{\pi(1)}{4}$, then for any ϕ there is only one equilibrium: in equilibrium, only one message is sent by each active sender. If $\gamma < \frac{\pi(1)}{4}$ there exists some ϕ for which an equilibrium exists with more than one message being sent by some senders.*

Proof: Recall first from (9) that if $\pi(1) \leq \frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]}$ then at most one message will be sent (the sender $\theta = 1$ is just indifferent to sending a further message if this holds with equality). Define $X = \frac{\gamma}{\pi(\theta_1)}$, and so this condition is

$$X[1 - X] \leq \frac{\gamma}{\pi(1)}.$$

The LHS is maximized, at $X = 0.25$. Therefore, all senders will send at most one message if $\gamma \geq \frac{\pi(1)}{4}$ (and there will be some senders if $\gamma < \pi(1)$). By Corollary 1, there can only be one equilibrium for any ϕ. If $\gamma < \frac{\pi(1)}{4}$ then (9) implies that $\Phi(\theta_1)$ involves $k = 2$ for some θ_1. Choosing the corresponding value of ϕ suffices to sustain such θ_1 as an equilibrium.

Q.E.D.

In the case of a linear profit function, the equilibrium involves one message per sender (with some active senders) if $\gamma \in \left(\frac{\bar{\pi}}{4}, \bar{\pi}\right)$.

3 Conditional examination functions

To describe multiple equilibria, it is helpful to define the construction of the function $\Phi(\theta_1)$ from its component pieces. In particular, we will show that the function $\Phi(\theta_1)$ is the upper envelope of functions $\Phi_i(\theta_1)$ where each of these sub-functions is defined by for a fixed k. In terms of the 6-step procedure described above, the functions $\Phi_i(\theta_1)$ are given by skipping step 4. The functions $\Phi_i(\theta_1)$ can therefore be viewed as those corresponding to a constrained problem, where at most i messages are allowed to be sent by each sender.

We define $\Phi_i(\theta_1)$ by modifying (13):

$$\Phi_i(\theta_1) = \frac{n_i(\theta_1)}{\pi(\theta_1)}\gamma,$$

where we define (by restricting (10):

$$n_i(\theta_1) = M[1-\theta_1] + M[1-\theta_2] + ...M[1-\theta_i], \tag{14}$$

as the number of messages sent if senders are restricted to send at most i messages each. To find these numbers, we follow the algorithm given above to find $\Phi(.)$ except we no longer impose step 4 (in which we endogenously determined k). The domain of $\Phi_i(\theta_1)$ is the set of θ_1 values for which some senders want to send at least i messages. By Lemma 1, this is true if it holds for sender $\theta = 1$. Recall from (9) that a sufficient statistic for sender $\theta = 1$ to want to transmit i messages rather than any lower number is if it prefers i to $i-1$. Equivalently, the incremental profit from another message should exceed its marginal cost, γ, or

$$\pi(1)\left[1-\frac{\phi}{n_i(\theta_1)}\right]^{i-1}\frac{\phi}{n_i(\theta_1)} \geq \gamma. \tag{15}$$

The two terms on the LHS of this expression represents the probability that the i-th message connects given the previous $i-1$ have missed. Moreover, we know that for θ_1 to just be indifferent between transmitting and not, we have (see (7))

$$\pi(\theta_1)\frac{\phi}{n_i(\theta_1)} = \gamma.$$

Substituting (7) into (15), the condition defining the domain of $\Phi_i(\theta_1)$ is

$$\pi(1)\left[1-\frac{\gamma}{\pi(\theta_1)}\right]^{i-1} \geq \pi(\theta_1). \tag{16}$$

Taking logarithms implies

$$[i-1] \geq \frac{\ln \pi(\theta_1) - \ln \pi(1)}{\ln\left(1 - \frac{\gamma}{\pi(\theta_1)}\right)}.$$

Recalling the definition of $F(\theta_1) = \frac{\ln \pi(\theta_1) - \ln \pi(1)}{\ln\left(1 - \frac{\gamma}{\pi(\theta_1)}\right)}$, this just means that $[i-1] \geq F(\theta_1)$. Equivalently, the function F is below the step level $i-1$ (see Figure 4). Thus the domain of $\Phi_i(\theta_1)$ is the interval defined by (16), which is the convex interval between the two roots, $\hat{\theta}_i^l$ and $\hat{\theta}_i^u$ of Equation (16).

Hence $\Phi_i(.)$ is the number of messages that must be examined in order to sustain a (constrained) equilibrium with $[1-\theta_1]$ senders, under the restriction that at most i messages can be sent per sender.

Proposition 6 *The function $\Phi(\theta_1)$ is the upper envelope of the restricted functions $\Phi_i(\theta_1)$, $i = 1, ..., k$.*

Proof: Figure 1 and the surrounding analysis indicates that the domain of $\Phi_i(\theta_1)$ is strictly contained in the domain of $\Phi_{i-1}(\theta_1)$. In turn, the domain of $\Phi_{i-1}(\theta_1)$ is contained in the domain of $\Phi_{i-2}(\theta_1)$, and so on, down to the largest domain, which is that of $\Phi_1(\theta_1)$, i.e., $\left[\pi^{-1}(\gamma), 1\right]$, so that the domain of $\Phi_i(\theta_1)$ is contained in those of all $\Phi_j(\theta_1)$, $j = 1, ...i-1$. Moreover, returning to the expressions for $n_i(\theta_1)$ as given by (14) above, ($n_i(\theta_1) = M[1-\theta_1] + M[1-\theta_2] + ...M[1-\theta_i]$) then we know that $n_i(\theta_1) > n_{i-1}(\theta_1)$ because the former has an extra positive term, $M[1-\theta_i]$ which is necessarily positive because on the domain of $\Phi_i(\theta_1)$ we have $\theta_i < 1$. Then recalling the definition above that $\Phi_i(\theta_1) = \frac{n_i(\theta_1)}{\pi(\theta_1)}\gamma$, this means that $\Phi_i(\theta_1) > \Phi_{i-1}(\theta_1)$ on the domain of $\Phi_i(\theta_1)$ (and, by the same argument, $\Phi_i(\theta_1)$ exceeds all the lower $\Phi_j(\theta_1)$, $j < i$). The uppermost of all the $\Phi_i(\theta_1)$ is $\Phi_k(\theta_1)$, which is by construction coincident with the function $\Phi(\theta_1)$. Hence $\Phi(\theta_1)$ is the upper envelope of the functions $\Phi_i(\theta_1)$.

Q.E.D.

Proposition 7 *Consider a set of equilibria, for given ϕ, ranked from low to high θ_1 values. Then the corresponding values of k (the maximal number of messages per sender) are non-decreasing. Furthermore, there are more messages sent at higher θ_1 values, so that the amount of congestion is higher: there is more shouting to be heard.*

Proof: From (7) we have $n = \phi\pi(\theta_1)/\gamma$, so that higher θ_1 imply higher n since ϕ is given. Thus, there is a higher amount of congestion (more "shouting to be heard"). We now show that the corresponding k cannot decrease. Recall (8):

$$\pi(\theta_j) = \frac{\pi(\theta_1)}{\left[1 - \frac{\gamma}{\pi(\theta_1)}\right]^{j-1}}.$$

Note that $\frac{d\theta_j}{d\theta_1}$ has the same sign as $\left[1 - \frac{j\gamma}{\pi(\theta_1)}\right]$. Hence, if $\frac{d\theta_j}{d\theta_1} > 0$, then $\frac{d\theta_i}{d\theta_1} > 0$ for all $i < j$. However, if $\frac{d\theta_k}{d\theta_1} > 0$, all the values of θ_j have increased and there must be fewer messages sent for the higher level of θ_1, a contradiction. Hence it must be that $\frac{d\theta_k}{d\theta_1} < 0$, and at least as many message levels must be present to generate a larger message volume at higher θ_1. Q.E.D.

This means that either there is the same number of levels of message-sending (same k) as we consider equilibria with higher θ_1, and the kth level kicks in earlier (i.e., at a lower level of θ); or else the final step is bigger than before. In both cases, this is needed to get the higher number of messages sent.

Message transmission may involve an intermediary (or intermediaries) such as the Post Office which is in charge of actual delivery, or billboard owners who need to display messages. The consequences for the revenues of such a transmitting intermediary are that the equilibria with higher θ_1 involve higher revenues. They involve higher profits if the intermediary sets a mark-up and produces at constant cost.

The next result follows from Proposition 7 and the property that equilibrium profits per sender decrease in n (see (3) and applying the envelope theorem); the aggregate result then follows directly.

Corollary 2 *Consider a set of equilibria, for given ϕ, ranked from low to high θ_1 values. Profits of each active sender are strictly decreasing across equilibria, as too are aggregate profits.*

Figure 5 gives the Φ_i functions for the linear profit example with $\bar{\pi}/\gamma = 20$: recall that $\Phi(\theta_1)$ is the upper envelope of these functions.[6]

[6] Similarly, in Figure 2, the upper envelope is the function Φ. The black curve is Φ_1, the red one is Φ_2, the green one is Φ_3, the sienna one is Φ_4.

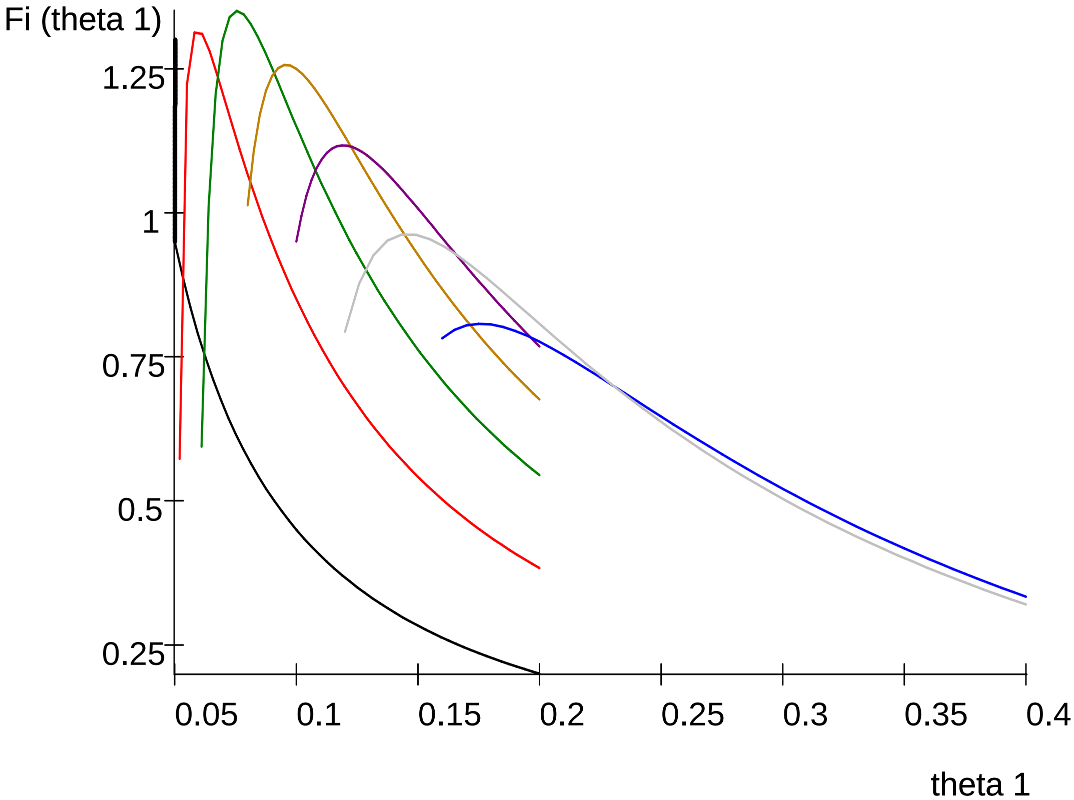


Figure 5. The functions Φ_i and their upper envelope.

The linear example, $\pi(\theta) = \bar{\pi}\theta$, affords some further characterization results.

Proposition 8 *Assume that* $\pi(\theta) = \bar{\pi}\theta$*. For any* $\theta \in (\gamma/\bar{\pi}, 1]$ *the examination value* $\Phi(\theta_1)$ *is piecewise quasi-concave: the functions* $\Phi_i(\theta_1)$ *are quasi-concave.*

Proof: Note first that

$$\Phi_i(\theta_1) = \frac{n_i(\theta_1)\gamma}{\bar{\pi}\theta_1}.$$

Hence:

$$\frac{\bar{\pi}}{\gamma}\frac{d\Phi_i}{d\theta_1} = \frac{n_i'(\theta_1)\theta_1 - n_i(\theta_1)}{[\theta_1]^2},$$

where, for the linear formulation, we have:

$$n_i(\theta_1) = i - \theta_1 \sum_{j=1\ldots i} \frac{1}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)^{j-1}}.$$

On the interior of the domain of $\Phi_i(\theta_1)$, we have:

$$n_i'(\theta_1) = -\sum_{j=1\ldots i} \frac{1 - \frac{\gamma}{\bar{\pi}\theta_1} j}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)^{j}}.$$

Hence,

$$\frac{\bar{\pi}}{\gamma} [\theta_1]^2 \frac{d\Phi_i(\theta_1)}{d\theta_1} = -i + \frac{\gamma}{\bar{\pi}} \sum_{j=1\ldots i} \frac{j-1}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)^{j}} = \Omega_i. \tag{17}$$

This expression implies that the function $\Phi_i(\theta_1)$ is quasi-concave since the factor on the LHS is positive and so $\frac{d\Phi_i}{d\theta_1}$ has the sign of the RHS. The RHS is decreasing in θ_1: this implies quasi-concavity because $\frac{d\Phi_i}{d\theta_1}$ is either of the same sign throughout its range so Φ_i is always decreasing (or increasing) or else it switches sign from positive to negative (so that Φ_i is increasing and then decreasing).

Q.E.D.

Example. In regime $k = 1$ we have that $\frac{d\Phi}{d\theta_1} < 0$ (see Corollary 1). In regime $k = 2$, we have:

$$\Omega_2 = -2 + \frac{\gamma/\bar{\pi}}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)^2}.$$

We compute this value at the critical point $\theta_2 = 1$. Recall that : $\theta_2 = \frac{\theta_1}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)}$ which is equal to 1 in this case. Therefore:

$$\Omega_2 = -2 + \frac{\gamma/\bar{\pi}}{[\theta_1]^2},$$

and so $\Omega_2 > 0$ if and only if $\theta_1 < \sqrt{\gamma/2\bar{\pi}}$.

Since the critical value of $\theta_2 = \frac{\theta_1}{\left(1 - \frac{\gamma}{\bar{\pi}\theta_1}\right)} = 1$, we solve this equation for its 2 solutions and choose the lower one (since the transition is from the regime with $k = 1$ to that with $k = 2$, i.e., the solution is the lower one solving $\theta_1^2 - \theta_1 + \frac{\gamma}{\bar{\pi}} = 0$, or $\theta_1 = \left(1 - \sqrt{1 - 4\frac{\gamma}{\bar{\pi}}}\right)/2$. The condition $\theta_1 < \sqrt{\gamma/2\bar{\pi}}$ now reduces to

$$1 - 3\frac{\gamma}{\bar{\pi}} < \sqrt{1 - 4\frac{\gamma}{\bar{\pi}}}$$

or

$$1 + 9\left(\frac{\gamma}{\bar{\pi}}\right)^2 - 6\frac{\gamma}{\bar{\pi}} < 1 - 4\frac{\gamma}{\bar{\pi}}$$

which is satisfied if $\frac{\gamma}{\pi} < \frac{2}{9}$. In this case, there is multiplicity of solutions for some values of ϕ. A necessary condition is therefore that $\theta_1 < 1/3$.

4 Comparing equilibria

For the normative analysis we approximate the step function for the number of messages sent with the function that generates them, namely $\pi(\theta_j) = \frac{\pi(\theta_1)}{\left(1-\frac{\gamma}{\pi(\theta_1)}\right)^{j-1}}$ ((8) above).[7] This has the advantage of matching the step levels at the integers ($\theta = \theta_j$), but it overstates the step representation otherwise (in between steps). We thus treat j as a continuous variable; we can invert the function (8) to give

$$j(\theta, \theta_1) = \frac{\ln[\pi(\theta_1)/\pi(\theta)]}{\ln[1-\gamma/\pi(\theta_1)]} + 1 \tag{18}$$

with $j(\theta_j, \theta_1) = \theta_j$, and $j(\theta, \theta_1) \in (0, k+1)$,[8] $j\left(\hat{\theta}, \theta_1\right) = 0$, i.e., $\hat{\theta}$ satisfies $\pi\left(\hat{\theta}\right) = \pi(\theta_1) - \gamma$ (and hence the value $\hat{\theta}$ is increasing in θ_1). In what follows, we shall further abuse the notation to write the function in (18) as $j(\theta)$ and use subscripts to indicate the particular θ_1 that pins it down.

We showed in Corollary 2 that all active senders' profits are higher at an equilibrium with a lower θ_1. As we shall show next, it is also possible to rank the solutions by their consumer surplus and total surplus.[9] We proceed via a series of Lemmata. We break down a comparison of two equilibria into a numbers effect and an allocation effect. We first show (Lemma 6) that consumer surplus and social surplus are both enhanced by a proportional decrease in any equilibrium message density. Bearing this in mind, we then work with normalized densities (where we reduce one to be number-equivalent to another; i.e., having the same number of total messages). We then show (Lemma 7) that these normalized densities retain a single crossing property, and that the density with the lower $\hat{\theta}_1$ has the higher consumer surplus and social surplus (Lemma 8). Proposition 9 then pulls these results together to show that equilibria are Pareto ranked for senders and consumers: lower total transmission rates correspond to better equilibria.[10] Proposition 6 then draws on this analysis to give the impact of increasing γ on various surpluses. Suppose that the receiver's

[7] For the linear profit function case, this is $\theta_j = \frac{\theta_1}{\left(1-\frac{\gamma}{\theta_1}\right)^{j-1}}$ or $j = 1 + \frac{\ln\frac{\theta_1}{\theta_j}}{\ln\left(1-\frac{\gamma}{\theta_1}\right)}$.

[8] More precisely, the exact upper bound is $\frac{\ln[\pi(\theta_1)/\pi(1)]}{\ln[1-\gamma/\pi(\theta_1)]} + 1$.

[9] If there is also an intermediary delivering the messages (the Post Office, for example), then its revenues are higher at higher-θ_1 equilibria. Its profits are higher too if it prices above (constant) marginal social delivery cost. This indicates the intermediary's incentives may run counter to those of the other agents.

[10] A related property is shown in the Appendix working with the constrained-optimal functions for a continuous message choice problem (Lemma 9).

surplus from a sender of type θ is $s(\theta)$.

Lemma 6 *Consider a message density, $l(\theta)$ with congestion; i.e., $\int_0^1 l(\theta)\,d\theta = n > \phi$. Then a proportional increase in the message density reduces consumer surplus and gross profit.*

Proof: Consumer surplus is given as $\int_0^1 s(\theta)\left\{1-\left(1-\frac{\phi}{n}\right)^{l(\theta)}\right\}d\theta$ (since the term in the parentheses is the probability of getting at least one message of type θ). Suppose now that all messages are duplicated β-fold, so the total number sent also goes up by the same factor, β. Then consumer surplus is $S(\beta) = \int_0^1 s(\theta)\left\{1-\left(1-\frac{\phi}{\beta n}\right)^{\beta l(\theta)}\right\}d\theta$. We show that $\left(1-\frac{\phi}{\beta n}\right)^{\beta l(\theta)}$ is increasing in β, which will prove the result. Equivalently, we wish to show that $\beta l(\theta)\ln\left(1-\frac{\phi}{\beta n}\right)$ is increasing in β: i.e., $l(\theta)\left\{\ln(1-x)+\frac{x}{1-x}\right\} > 0$ where $x = \frac{\phi}{\beta n} > 0$. Since the bracketed term is increasing in x, and is zero at $x=0$, the result follows directly. The gross profit result follows by the same argument: aggregate gross profit is simply $\int_0^1 \pi(\theta)\left\{1-\left(1-\frac{\phi}{n}\right)^{l(\theta)}\right\}d\theta$.
Q.E.D.

Intuitively, more transmission density for each type implies greater duplication since the likelihood of receiving multiple messages of the same type (which just serves to crowd out other messages) increases.[11] With this result in mind, we can now consider normalized densities such that the mass of messages sent is standardized.

Definition 1 *Consider two transmission densities with n_A and n_B the associated numbers of messages sent. Then $\tilde{l}_B(\theta) = \frac{n_A}{n_B} l_B(\theta)$ is the normalized (or number-adjusted) second density.*

The normalized equilibrium densities are compared in the next Lemma.

Lemma 7 *Consider two equilibrium transmission densities $l_A(\theta)$ and $l_B(\theta)$. Suppose they are both monotone increasing, with $\hat{\theta}_A < \hat{\theta}_B$, and with $n_B \geq n_A \geq \phi$ as the corresponding numbers of messages sent. Then there exists a unique value θ^I such that $l_A\left(\theta^I\right) = \tilde{l}_B\left(\theta^I\right)$: i.e., the number-adjusted densities cross only once.*

Proof: Consider the slope of any equilibrium transmission density, $j_i(\theta)$. Since $j_i(\theta) = \frac{\ln[\pi(\theta_1)/\pi(\theta)]}{\ln[1-\gamma/\pi(\theta_1)]} + 1$ by

[11] Note that this result does NOT imply that the optimal message density is zero: recall that we need the constraint $\bar{\phi} < \beta n$ for the expressions given to be correct (i.e., involve congestion). This would clearly be violated if β were too low.

(18), the slope expression is

$$\frac{dj_i(\theta)}{d\theta} = \frac{-\pi'(\theta)}{\pi(\theta)} \frac{1}{\ln\left[1-\gamma/\pi\left(\theta_1^i\right)\right]}.$$

This implies that the density slope is always lower for the density with fewer messages (lower θ_1 or $\hat{\theta}$). However, the comparison between the slope of $\tilde{j}_B(\theta)$ and $j_A(\theta)$ is more intricate because the weights flatten the relationship:

$$\frac{d\tilde{j}_B(\theta)}{d\theta} = \frac{n_A}{n_B} \frac{-\pi'(\theta)}{\pi(\theta)} \frac{1}{\ln\left[1-\gamma/\pi\left(\theta_1^B\right)\right]}. \tag{19}$$

At any crossing point, θ^I, $j_A\left(\theta^I\right) = \tilde{j}_B\left(\theta^I\right)$, so $j_A\left(\theta^I\right) = \frac{n_A}{n_B} j_B\left(\theta^I\right)$, from which we derive the ratio relation that must hold as

$$\frac{n_A}{n_B} = \frac{\frac{\ln\left[\pi\left(\theta_1^A\right)-\gamma\right]-\ln\pi\left(\theta^I\right)}{\ln\left[\pi\left(\theta_1^A\right)-\gamma\right]-\ln\pi\left(\theta_1^A\right)}}{\frac{\ln\left[\pi\left(\theta_1^B\right)-\gamma\right]-\ln\pi\left(\theta^I\right)}{\ln\left[\pi\left(\theta_1^B\right)-\gamma\right]-\ln\pi\left(\theta_1^B\right)}}.$$

Using this expression in (19) gives

$$\begin{aligned} \frac{dj_A\left(\theta^I\right)}{d\theta} &< \frac{d\tilde{j}_B\left(\theta^I\right)}{d\theta} \quad \text{as} \\ \frac{1}{\ln\left[1-\gamma/\pi\left(\theta_1^A\right)\right]} &> \frac{1}{\ln\left[1-\gamma/\pi\left(\theta_1^B\right)\right]} \frac{\frac{\ln\left[\pi\left(\theta_1^A\right)-\gamma\right]-\ln\pi\left(\theta^I\right)}{\ln\left[\pi\left(\theta_1^A\right)-\gamma\right]-\ln\pi\left(\theta_1^A\right)}}{\frac{\ln\left[\pi\left(\theta_1^B\right)-\gamma\right]-\ln\pi\left(\theta^I\right)}{\ln\left[\pi\left(\theta_1^B\right)-\gamma\right]-\ln\pi\left(\theta_1^B\right)}}, \text{ or} \\ \ln\left[\pi\left(\theta_1^B\right)-\gamma\right] &> \ln\left[\pi\left(\theta_1^A\right)-\gamma\right], \end{aligned}$$

which is true since $\theta_1^B > \theta_1^A$. Hence, $\frac{\tilde{j}_B'(\theta)}{j_A'(\theta)} > 1$ and so, since $j_i(\theta)$ is a continuous function, there can be only one crossing.

Q.E.D.

We now compare surpluses for the number-adjusted densities. We shall assume that consumer surplus rises no faster than profit with θ :

Assumption 5: $\frac{s(\theta_\beta)}{s(\theta_\alpha)} \leq \frac{\pi(\theta_\beta)}{\pi(\theta_\alpha)}$ for all $\theta_\alpha < \theta_\beta$.

This assumption means that there will be a bias toward the high profit types in equilibrium.[12]

Lemma 8 *Consider two equilibrium numbers-adjusted transmission densities $j_A(\theta)$ and $\tilde{j}_B(\theta)$, with $\hat{\theta}_A < \hat{\theta}_B$. Assume that $\frac{s(\theta_\beta)}{s(\theta_\alpha)} \leq \frac{\pi(\theta_\beta)}{\pi(\theta_\alpha)}$ for all $\theta_\alpha < \theta_\beta$. Then both consumer surplus and total surplus are greater under density $j_A(\theta)$ than under density $\tilde{j}_B(\theta)$.*

[12] The Assumption corresponds to the elasticity restriction considered in the Appendix.

Proof: First consider the graphs of $j_A(\theta)$ and $\tilde{j}_B(\theta) = \frac{n_A}{n_B} j_B(\theta)$ in (j, θ) space, where the factor $\frac{n_A}{n_B}$ serves to scale down the density j_B so that its integral is n_A. The graph can be split into 3 Regions: where the densities overlap, and two where one is present and the other absent. Denote by Region 1 the area of the graph where density A is present without density B, and Region 2 the area of the graph where density B is present without density A. By construction, Regions 1 and 2 have the same area. Suppose we take some of the density from Region 1 and place it in Region 2 (ultimately, we shall place it all there).

We start from the message density $j_A(\theta)$ and transfer mass in small pieces in order to reproduce density $\tilde{j}_B(\theta)$. So, first take away a small set of messages at θ_α in Region 1, of mass Δ_j, and add an equal mass at θ_β in Region 2. This change leaves the total number of messages the same, but reduces profits of sender types at both θ_α and θ_β since $j_A(\theta)$ was derived as the profit maximizing transmission level for the given number of total messages (n_A). Recall that the profit of a sender of type θ is $\pi(\theta)\mathbb{P}(j,n) - \gamma j$, where $\mathbb{P}(j,n)$ is the probability that at least one of sender θ's messages is examined (and here we have $\mathbb{P}(j,n) = 1 - \left(1 - \frac{\phi}{n}\right)^j$). Hence, the change in profit at θ_α is $\pi(\theta_\alpha)\Delta_{\mathbb{P}_\alpha} + \gamma\Delta_j$, where $\Delta_{\mathbb{P}_\alpha} < 0$ is the reduction in the probability of examining at least one message from type θ_α. Note that the change in profit is negative for each further marginal reduction in the density at θ_α as long as $\mathbb{P}(j,n)$ is a concave function of j (which property holds here).

Similarly, the change in profit at θ_β is $\pi(\theta_\beta)\Delta_{\mathbb{P}_\beta} - \gamma\Delta_j$ where $\Delta_{\mathbb{P}_\beta} > 0$. Both profit changes are negative since $j_A(\theta)$ represented the profit-maximizing density – at θ_α there are now fewer messages than maximizes profits, while at θ_β there are now too many.

Hence, for all such transfers of message mass, the inequality $\pi(\theta_\alpha)\Delta_{\mathbb{P}_\alpha} + \pi(\theta_\beta)\Delta_{\mathbb{P}_\beta} < 0$ holds. Equivalently, $\frac{\pi(\theta_\beta)}{\pi(\theta_\alpha)} < \frac{-\Delta_{\mathbb{P}_\beta}}{\Delta_{\mathbb{P}_\alpha}}$: since we assume that $\frac{s(\theta_\beta)}{s(\theta_\alpha)} \leq \frac{\pi(\theta_\beta)}{\pi(\theta_\alpha)}$, this implies that $\frac{s(\theta_\beta)}{s(\theta_\alpha)} < \frac{-\Delta_{\mathbb{P}_\beta}}{\Delta_{\mathbb{P}_\alpha}}$: rearranging gives $s(\theta_\alpha)\Delta_{\mathbb{P}_\alpha} + s(\theta_\beta)\Delta_{\mathbb{P}_\beta} < 0$ which implies that the consumer surplus must go down. As we already showed that profits go down, total welfare must go down.

We move mass such that at the end of this construction, we have moved all the mass of Region 1 to Region 2. As just noted, such a move reduces consumer surplus and welfare.

Q.E.D.

We now add further messages (of mass $n_B - n_A$) in proportion to the derived density, $\frac{n_A}{n_B} j_B(\theta)$ in order to arrive at $\tilde{j}_B(\theta)$. By Lemma 6, this transformation further reduces consumer surplus. Therefore the total effect is to reduce consumer surplus (because of the increased redundancy). Adding the second set of messages causes consumer surplus to fall because it increases duplication on those messages that are already overrepresented in the population of messages transmitted.[13] The next Proposition sums up.

Proposition 9 *Suppose senders can transmit multiple messages and assume that* $\frac{s(\theta_\beta)}{s(\theta_\alpha)} \leq \frac{\pi(\theta_\beta)}{\pi(\theta_\alpha)}$ *for all* $\theta_\alpha < \theta_\beta$*. Then, any two equilibrium solutions as indexed by* $\hat{\theta}_A < \hat{\theta}_B$ *can be Pareto ranked: no sender is worse off at A than at B, and active senders are better off; consumers are better off.*

Proof: It was argued in Proposition 7 that if $\hat{\theta}_A < \hat{\theta}_B$, then $n_A < n_B$, and profits are higher for active senders at the lower solution. So consider the consumer. By Lemmata 6 through 8 above, we need simply show that the transmission density functions corresponding to the two equilibria exhibit the property that $j_A(\theta)$ crosses $j_B(\theta)$ from above. We know that both $j_A(\theta)$ and $j_B(\theta)$ are increasing and continuous with $\hat{\theta}_A < \hat{\theta}_B$. Since $n_A < n_B$, $j_A(\theta)$ and $j_B(\theta)$ must cross at least once. At any such crossing, we know from the proof of Lemma 7 that the slope of the former is less than that of the latter, so proving the point.
Q.E.D.

As noted above, it may though be true that the intermediary in charge of delivering messages (if there is one) is better off at B than at A if the intermediary prices above marginal social cost.

5 Raising the cost of sending messages

The primary instrument for effecting welfare changes is the transmission cost. This variable is set by the Post Office (or the relevant regulatory commission) in the case of Bulk Mailing, and can be affected by the tax treatment of advertising in other cases. For congestion with multiple message sending (which engenders obvious duplication) and multiple equilibria, it is important to determine how the equilibria change, and who may gain or lose. We consider three perspective angles on the effects of changes in the cost of sending messages.

The first raises the transmission rate to keep the marginal sender the same, but reduces the number messages to one per sender. The second is the First-Best optimum allocation. The third is a local change in the transmission cost.

[13] The first best optimum involves one message only being sent per type θ, with the total number of messages being $\bar{\phi}$, so there is no congestion. As long as $b = s + \pi$ is increasing, these are the "top" $\bar{\phi}$ messages.

In what follows, we will make use of the following property, which we recall from Lemma 2(ii). Namely, a marginal sender type $\theta_1 = \pi^{-1}(\gamma)$ can be supported as an equilibrium if and only if $\phi \geq [1-\theta_1] M$, and that such an equilibrium entails a transmission cost $\gamma = \pi(\theta_1)$ and $k = 1$, i.e., a single message per sender. In terms of Figure 6 below, the upper envelope function, $\Phi(\theta_1)$ is vertical at $\pi^{-1}(\gamma)$ down to the dotted green line which represents the equation $\phi \geq [1-\theta_1] M$ [14], and higher values of γ correspond to functions $\Phi(\theta_1)$ to the right (the particular function drawn sets $\gamma = 0.1$, and the vertical segment corresponds to the vertical axis above $\phi = 0.9$). The importance of the line $\phi = [1-\theta_1] M$ is that any (ϕ, θ_1) below it must have congestion, even if senders sent only one message. Above or on it, there would be no congestion if senders only sent one message.[15]

5.1 Keeping the marginal sender constant

The experiment here is to consider any equilibrium θ_1 with $n > \phi$ (so that there is congestion initially), as induced by γ and ϕ, and then to raise γ (if necessary) so as to reach an equilibrium with the same θ_1 (i.e., the same senders active) but with only one message sent per sender and no congestion. As per Lemma 2(ii), an equilibrium with just one message per sender can be achieved through appropriate choice of γ if and only if $[1-\theta_1] M \leq \phi$ for the original θ_1 (where we recall that $[1-\theta_1] M$ is the mass of types sending).[16] Such an equilibrium can be attained if $[1-\theta_1] M \leq \phi$ by choosing a transmission rate $\tilde{\gamma} = \pi(\theta_1)$. This entails an increase in the transmission cost because originally we had $\gamma = \frac{\phi}{n}\pi(\theta_1)$ determining θ_1, with $n > \phi$, (i.e., congestion): at such a transmission cost, with one message per sender, the number of messages sent is $n = [1-\theta_1] M < \phi$. Such an equilibrium corresponds to the vertical axis above the black line in Figure 2, 3 or 5.

Proposition 10 *Consider a congested equilibrium with marginal sender type θ_1. If $[1-\theta_1] M > \phi$, it is not possible by choice of a transmission cost to eliminate congestion and retain all senders active. Otherwise, for $[1-\theta_1] M \leq \phi$, there exists a cost $\hat{\gamma} > \gamma$ such the receiver and almost all active senders are better off with the higher rate (even if transmission revenues are discarded).*

[14] The black line, for one message per firm, is always below the green line because the black line involves congestion.

[15] Along the locus $\phi = [1-\theta_1] M$, the revenue of the delivery intermediary is increasing or decreasing according to the sign of the derivative of $[1-\theta_1]\pi(\theta_1)$. For the linear case, the revenue is simply $\bar{\pi}\theta_1(1-\theta_1)$, analogous to a linear demand curve.

[16] If $[1-\theta_1] M > \phi$, then, even if we could induce only one message from each sender, there would still be congestion because $[1-\theta_1] M = n > \phi$.

Proof. Given the discussion preceding the Proposition, it remains to prove that all agents are no worse off given $\tilde{\gamma}$ when $[1-\theta_1] M \leq \phi$. The receiver is better off (at an equilibrium where she now receives messages from the same senders, but now with only one message per sender) because duplication is eliminated, and she now examines all the messages sent.

It remains to consider the senders. Clearly the marginal sender is indifferent between the original cost γ and the higher cost $\hat{\gamma}$ because it makes zero profit before and after. We now show that profits are higher for all senders $\theta > \theta_1$; equivalently, we establish that the following relation holds:

$$\pi(\theta) - \hat{\gamma} > \pi(\theta)\left[1 - \left(1 - \frac{\phi}{n}\right)^{l^*}\right] - l^*\gamma \qquad \text{for all } \theta > \theta_1$$

where l^* denotes the number of messages sent by a sender of type θ at the original equilibrium. The LHS of this expression is the profit per sender at the new equilibrium (when its message is examined for sure, and it sends only one, at cost $\hat{\gamma}$); the RHS is its original profit. We also know that, by construction, $\pi(\theta_1) = \hat{\gamma}$ and $\pi(\theta_1)\frac{\phi}{n} = \gamma$, so that $\hat{\gamma} = \frac{n}{\phi}\gamma$. After replacing this expression, we therefore want to show that

$$\pi(\theta) - \hat{\gamma} > \pi(\theta)\left[1 - \left(1 - \frac{\phi}{n}\right)^{l^*}\right] - l^*\frac{\phi}{n}\hat{\gamma},$$

or

$$\pi(\theta)\left(1 - \frac{\phi}{n}\right)^{l^*} > \left[1 - l^*\frac{\phi}{n}\right]\hat{\gamma},$$

which is true for $\theta > \theta_1$ since $\pi(\theta) > \pi(\theta_1) = \hat{\gamma}$ because the inequality:

$$\left(1 - \frac{\phi}{n}\right)^{l^*} > \left[1 - l^*\frac{\phi}{n}\right]$$

holds strictly for $l^* > 1$.

Q.E.D.

Note that the receiver would examine more messages than the number actually sent, and so further welfare gains could be realized. It is also striking that profits from higher transmission costs need not be redistributed in order to make everyone better off: this is supposing that transmission was originally priced at cost and is then raised above cost. It is noteworthy that the delivery intermediary's revenues actually fall. Higher prices are more than offset by considerably lower volume.

Figure 6 repeats the envelope Φ from Figure 5, except now with the "congestion" locus $\phi = [1 - \theta_1] M$ drawn in light green. The experiment of this sub-section takes an equilibrium point above this line, and effectively drops it down to the line by raising γ.

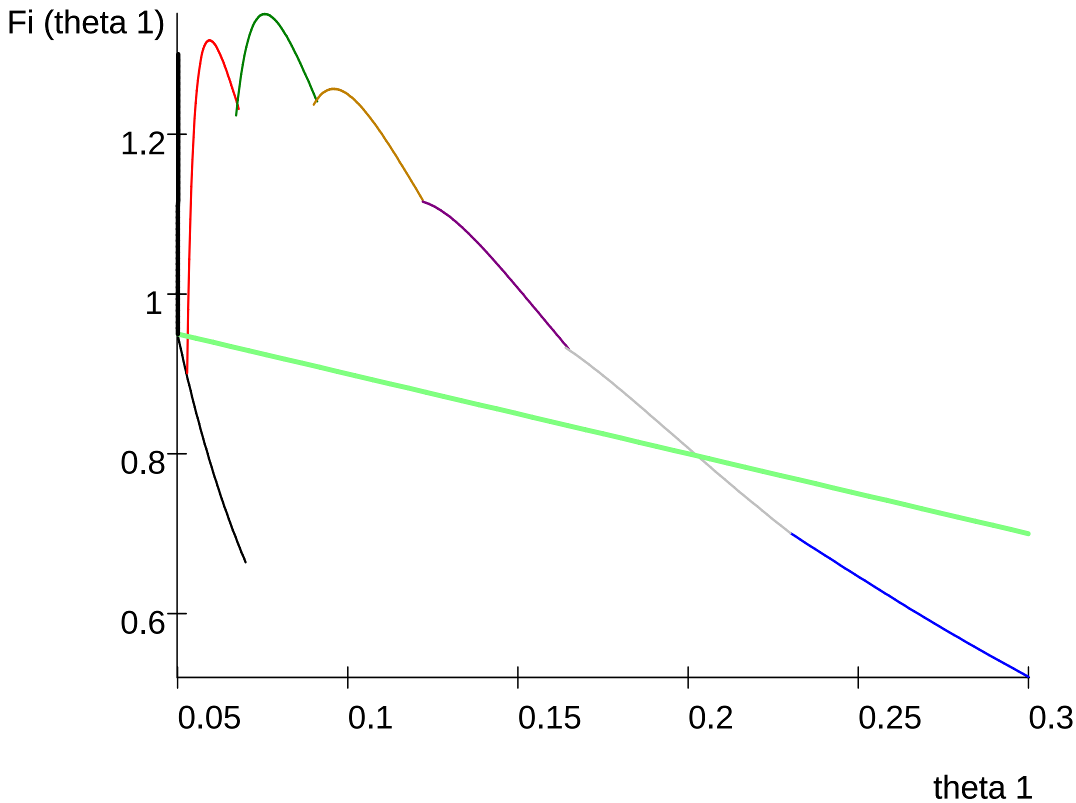


Figure 6. The upper envelope of the functions Φ_i and $\Phi = 1 - \theta_1$.

5.2 First-best optimum

The first best optimal allocation has all messages examined, with one sent per sender, and only the top θ ones are sent, so the optimal cut-off, θ_1^o, is determined from $[1 - \theta_1^o] M = \phi$. From the previous sub-section, such an allocation can be decentralized as an equilibrium by appropriate choice of γ (although there may be other equilibria for the same transmission cost). In terms of Figure 6, the first-best optimum is a point on the green line for the given level of ϕ. As we show below, the market solution can involve too many or too few active senders. We here determine how the welfare of the market participants changes moving from an equilibrium to the (decentralized) first-best (and so correspond to horizontal comparisons in Figure 6).

The first best has $n^o = [1 - \theta_1^o] M = \phi$, and the value of γ that can yield this as an equilibrium (with one message per sender) is $\gamma^o = \pi(\theta_1^o)$. This marginal sender type can be above or below the equilibrium level if there is congestion,[17] so we consider both cases. First, suppose that the original equilibrium has a lower marginal sender, which means increasing the transmission cost.[18] Then, at least some senders are worse off (for starters, those now excluded). The receiver is better off: there is no duplication, and the worst messages have been censored. She only gets the top ones, and every one is a hit. The Post Office's revenues would rise: revenues before are $n\gamma = \phi\pi(\theta_1)$ (simply by rearranging (7)) and after they are $\phi\gamma^o = \phi\pi(\theta_1^o)$, and the latter is higher.

Second, consider an equilibrium with a higher marginal sender ($\theta_1^o < \theta_1$). Again, the receiver is better off, now because she gets messages from more senders, and all are hits. Equilibrium entailed both a limited range of messages, and duplication too. An equilibrium with a higher marginal sender is consistent with either a higher transmission cost or a lower one than γ^o.[19]

In the former case ($\gamma^o > \gamma$), γ is raised but the marginal sender falls. The new marginal sender, previously excluded, makes zero profits, but every new sender type now included (those between $\pi^{-1}(\gamma^o)$ and the original θ_1) is now better off, with a positive profit. Likewise, with an argument following the proof of Proposition 10, all higher sender types are better off too. Therefore, no sender is worse off, and all active ones (bar the new marginal one) are strictly better off. Finally, consider $\gamma > \gamma^o$ and $\theta_1^o < \theta_1$. More senders are active, and all active senders are better off: only one message is sent, which always gets through, and at a lower price.

In summary:

Proposition 11 *The first-best optimum, $\theta_1^o = \left[1 - \frac{\phi}{M}\right]$, can be decentralized as an equilibrium by setting $\gamma^o = \pi(\theta_1^o)$. The receiver is always better off. If $\theta_1 > \theta_1^o$, no sender is worse off even if the cost of sending*

[17] To see this, recall that the Φ curve is not a function of the actual ϕ. In Figure 6, for example, the intersection of a given level of ϕ with the envelope curve Φ can lie either side of the locus $\phi = [1 - \theta_1] M$.

[18] This follows because the original equilibrium has $\frac{\phi}{n}\pi(\theta_1) = \gamma$, with $\phi < n$, and the decentralized optimum involves $\pi(\theta_1^o) = \gamma^o$ with $\theta_1 < \theta_1^o$. Hence $\gamma^o > \frac{n}{\phi}\gamma > \gamma$.

[19] For example, imagine a horizontal line representing ϕ in Figure 6 for some ϕ just below 0.95, which is the vertical intercept of the green line. Then $\theta_1^o < \theta_1$, and $\gamma^o > \gamma$. With a ϕ around 0.97, say, the congested equilibrium on the red or purple curves has a higher marginal sender, $\theta_1^o < \theta_1$, but γ^o is lower than the transmission cost, γ, drawn in the Figure.

messages rises.

The final part of the Proposition reflects the earlier finding of strong inefficiency of equilibria above the congestion locus.

5.3 Local changes in transmission rate

We here look at local changes around any equilibrium point, without immediate concern for the possibility that the equilibrium might jump (so we assume that the maximal number of messages sent remains the same). Given this focus, we consider a (local) rise in γ. We assume that the situation remains congested (i.e. we retain $\pi(\theta_1) > \gamma$).

When γ rises, the RHS of the curve in (11) falls for any given value of k. This means that the total number of messages, n, falls *for given* θ_1. However, the change in the curve $\Phi(\theta_1)$ is ambiguous, so the final effect on θ_1 is ambiguous (and also depends on whether the curve $\Phi(\theta_1)$ was locally rising or locally falling).

We can break down the cases by looking at the elasticity of the number of messages, n, with respect to γ. Results go in opposite directions according to whether this elasticity is less than -1 or greater than 0.

Proposition 12 *Consider the effects of raising γ. If $n\gamma$ falls with γ, then θ_1 rises and n falls: consumer surplus goes up and profits rise for ALL active senders (even if the cost rise is not refunded). If $n\gamma$ rises with γ, then θ_1 falls: if n rises, consumer surplus goes down and profits fall for all senders (if the cost rise is not refunded).*

Proof: If $n\gamma$ falls, the elasticity of n with respect to γ is below -1. In this case, from (7), θ_1 falls locally in equilibrium when γ rises. We first argue that consumer surplus rises. This is not a priori obvious because there are now messages from low-surplus types, and the high-surplus ones (high θ) have less prominence.

The proof proceeds along the lines of that in Proposition 7. First, let the message density be $j_A(\theta)$ under the high sending cost, γ_A, and let it be $j_B(\theta)$ under the low sending cost, γ_B. The corresponding numbers of messages satisfy $n_A < n_B$. Define then the numbers-adjusted density, $\tilde{j}_B(\theta) = \frac{n_A}{n_B} j_B(\theta)$, as before. Then, consider as a starting point the equilibrium density $j_A(\theta)$ that is attained under cost γ_A. It generates maximal profit for each sender given the total volume of messages sent by other senders, and rearranging the sending density can only decrease aggregate profits, as Lemma 8 showed. Thus, aggregate profits are higher under $j_A(\theta)$ than under $\tilde{j}_B(\theta)$. Since revenues for any type, θ, are equal to the probability of a sale times $\pi(\theta)$, and consumer surplus enjoyed for any type is equal to the probability of a sale times $s(\theta)$, as long as $s(\theta)$ is proportional to $\pi(\theta)$ then consumer surplus is proportional to revenues, and so consumer surplus also goes down in a move from $j_A(\theta)$ to $\tilde{j}_B(\theta)$. This is a fortiori true when consumer surplus is less elastic than profit (as long as $j_A(\theta)$ rises more slowly than $\tilde{j}_B(\theta)$, as will be shown next.)

We now show the analogue to Lemma 7, namely that $j_A(\theta)$ and $\tilde{j}_B(\theta)$ cross only once, the latter from below. Following the steps of the proof of Lemma 7, the penultimate line in the proof becomes $\ln\left[\pi\left(\theta_1^B\right)-\gamma_B\right]>\ln\left[\pi\left(\theta_1^A\right)-\gamma_A\right]$ which has to be shown. But this must be true since both $\theta_1^B>\theta_1^A$ and $\gamma_A>\gamma_B$. Finally, the (duplication) argument of Lemma 6 indicates that consumer surplus is higher under $\tilde{j}_B(\theta)$ than $j_B(\theta)$. This completes the proof that consumer surplus is higher with a higher cost of sending.

We now show that all senders' profits are higher too. There are two conflicting forces here. First, direct cost γ has risen, but competition, n, has fallen (using the envelope theorem on profits in both cases). Clearly, since the support of senders has expanded, the low-θ types are better off since they now send while they chose not to send before. The key to the argument is that the competition effect dominates, and makes all better off. To see this, note from the condition (7) that $\phi\pi(\theta_1)=n\gamma$. Since θ_1 falls as γ rises, then $n\gamma$ falls (on the RHS). Writing this condition as a derivative, $\frac{d(n\gamma)}{d\gamma}<0$, we have

$$\frac{\gamma}{n}\frac{dn}{d\gamma}<-1. \tag{20}$$

Now consider the profit function (3): $\pi(\theta)\left[1-\left(1-\frac{\phi}{n}\right)^{\ell}\right]-\gamma\ell$. The derivative of the associated maximum value function with respect to γ is

$$-\pi(\theta)\,\ell\left(1-\frac{\phi}{n}\right)^{\ell-1}\frac{\phi}{n^2}\frac{dn}{d\gamma}-\ell,$$

which, noting that the choice of ℓ entails $\pi(\theta)\left(1-\frac{\phi}{n}\right)^{\ell-1}\left(\frac{\phi}{n}\right)>\gamma$ (see (4)), implies that $-\pi(\theta)\,\ell\left(1-\frac{\phi}{n}\right)^{\ell-1}\frac{\phi}{n^2}\frac{dn}{d\gamma}-\ell>\ell\left[-\frac{\gamma}{n}\frac{dn}{d\gamma}-1\right]$, which is positive by the elasticity condition (20) above. Thus all profits rise, as claimed.

If instead $n\gamma$ rises with γ, then from (7), θ_1 rises locally. A clean result obtains if n itself rises. Using the analogous arguments as used above, now applied in the opposite direction for a higher n, gives the final results in the Proposition.

Q.E.D.

6 Conclusions

The advantage to the sender from sending multiple messages is that this will improve the likelihood that the recipient does examine at least one of the sender's messages, and so raises the probability of a successful transaction. However, the recipient need only examine one message from a sender in order to make a transaction, so that any further messages received beyond the first one from a particular sender are wasted. This situation is formally analogous to a lottery with a fixed number of prizes, which are more highly valued by some (high θ types) than others. The chances of winning the lottery depend on the number of tickets bought, and the value of a further prize after winning once is zero (this is the assumption that the consumer need only examine one message from a sender to be informed of the opportunity). As we show, in equilibrium higher θ types send more ads. This is because they have higher profit from making a connection, and so they "buy more tickets."

We have modeled senders as choosing how many messages to send. Another way of breaking through the advertising clutter is for the sender to capture attention with the advertisement. The Marketing press stresses this strategy. In terms of our model, we could consider the variable l as choice of quality of advertising (which would naturally be a continuous variable). Here, higher "quality" is understood to entail higher break-through probability, and not (necessarily) intrinsic artistic merit. Our results suggest higher quality ads from senders with more to gain from a breakthrough; they also suggest a "race-to-the-top" where others are also trying to break through. Since this process engenders more effective clutter, this in turn suggests that spending caps (as in the political arena), or advertising limits or agreements (as with cigarettes) might be desirable. Such caps are already implicitly modeled through the Φ_i functions in the current set-up.

We took three perspectives on the normative economics of changing the transmission cost of messages. If the status quo involves high congestion (namely, above the locus $\phi = [1 - \theta_1] M$), the receiver and (almost) all active senders can be made better off simply by "standing still" and raising the transmission cost to retain the original set of active senders. Pricing out the duplicated messages makes even the senders better off despite a higher price per message. The Post Office's revenues would fall, which might constitute an incentive for it to resist such higher transmission prices.

We have kept the number of messages examined constant (at ϕ). Endogenous receiver effort levels are addressed in Anderson and de Palma (2007) for the case of a single message per firm. It is straightforward to see that the multiplicity of equilibria survives under endogenous examination (it suffices to take an examination function with a zero marginal opening cost below ϕ and a prohibitive one above it).

The issue of time is another interesting generalization. Everything in the current model is static, so that yesterday's decisions do not affect today's purchases. This may be viewed as daily receipt of junk mail, with consumers each day forgetting yesterday's offers. It may also be relevant to TV-watching with transient commercials that are already out of one's consciousness by the next break.

We have also assumed the independence of messages, in order to close down business-stealing effects within industries, but this could provide a potentially important caveat to the result that message cost rises improve welfare for all: one might expect competition within sectors to be reduced, which could harm welfare. In parallel research we are addressing this issue by considering sub-groupings of senders ("industries") that compete in the same market.

7 Appendix

Proof of Lemma 3. Existence of a unique solution is given by the 6-step procedure described above (see expressions (9), (11), and (13)). We first show that $n(\theta_1)$ is continuous. Define a regime by its corresponding value of k. The boundary between two regimes as the uppermost number of messages transmitted changes between k and $k-1$ is given by $\theta_k = 1$. Continuity of n with respect to θ_1 within each regime (i.e., for given k) follows from the fact that each of the 6 steps preserves continuity. Now consider the boundaries between regimes. In regime k with $\theta_k = 1$, we have (from (10)): $n = M[1-\theta_1] + M[1-\theta_2] + ...M[1-\theta_{k-1}]$, which is clearly equal to the expression for n in the regime $k-1$. Since the functions (8) defining θ_k as functions of θ_1 are continuous in θ_1, it is necessarily the case that continuity of $n(\theta_1)$ is preserved between neighboring regimes. Therefore n is also continuous across regimes. However, it is not generally differentiable at the critical points $\theta_1^c(k)$ corresponding to the boundary between k and $k-1$ (we will provide explicit expressions in the linear case). Differentiability of n in θ_1 within each regime follows because it is preserved across each of the six steps.

The last part is shown as follows. First we prove that $\lim_{\theta_1 \downarrow \pi^{-1}(\gamma)} \Phi(\theta_1) = M\left[1-\pi^{-1}(\gamma)\right]$. Let $\pi(\theta_1) = \gamma + \varepsilon$, $\varepsilon > 0$. Then we have the following inequalities from the bounds defining k (see (9)):

$$\frac{\gamma+\varepsilon}{\left[1-\frac{\gamma}{\gamma+\varepsilon}\right]^{k-1}} \leq \pi\left(1\right) < \frac{\gamma+\varepsilon}{\left[1-\frac{\gamma}{\gamma+\varepsilon}\right]^{k}},$$

for which the only solution for ε small enough is $k = 1$ (recall that $\pi\left(.\right)$ is continuous and that $\gamma < \pi(1)$). Given $k = 1$, then the number of messages sent tends to $n\left(\pi^{-1}\left(\gamma\right)\right) = M\left[1-\pi^{-1}\left(\gamma\right)\right]$. Furthermore, by (7), the congestion rate, n/ϕ, is unity (meaning all messages sent are examined). Thus ϕ must also tend to $M\left[1-\pi^{-1}\left(\gamma\right)\right]$, as was to be proven.

We next show that $\lim_{\theta_1 \uparrow 1} \Phi\left(\theta_1\right) = 0$. In this case, we have $\pi\left(\theta_1\right) = \pi\left(1\right) - \varepsilon$, for $\varepsilon > 0$. The inequalities defining k are now:

$$\frac{\pi\left(1\right)-\varepsilon}{\left[1-\frac{\gamma}{\pi(1)-\varepsilon}\right]^{k-1}} \leq \pi\left(1\right) < \frac{\pi\left(1\right)-\varepsilon}{\left[1-\frac{\gamma}{\pi(1)-\varepsilon}\right]^{k}},$$

which clearly has a unique solution $k = 1$ for ε small enough. Therefore $n\left(\theta_1\right) = M\left[1-\theta_1\right]$ and the number of messages sent tends to zero as $\theta_1 \uparrow 1$. Since $\phi < n$, ϕ must also tend to 0 as $\theta_1 \uparrow 1$, by (7).

Q.E.D.

7.1 Continuous messages

We here lay out the continuous version of the model so the number of messages sent per sender is a non-negative real number. Recalling that sender θ's problem (from (3)) is

$$\max_{\ell}\left\{\pi\left(\theta\right)\left[1-\left[1-\frac{\phi}{n}\right]^{\ell}\right]-\gamma\ell\right\}, \qquad \ell \geq 0,$$

we now seek the (continuous) solution to this maximization problem, and the corresponding equilibrium.

Optimization yields the first order condition

$$-\pi\left(\theta\right)\left[\left[1-\frac{\phi}{n}\right]^{\ell}\ln\left(1-\frac{\phi}{n}\right)\right] = \gamma, \tag{21}$$

where the L.H.S. is positive because the logarithm is negative. The L.H.S. is decreasing in l, indicating global concavity of the profit function.

Denote the solution to (21) $j\left(\theta\right) > 0$, so that:

$$j\left(\theta\right) = \frac{1}{\ln\left(1-\phi/n\right)}\ln\left(\frac{-\gamma}{\pi\left(\theta\right)\ln\left(1-\phi/n\right)}\right), \tag{22}$$

where both terms are negative on the R.H.S. (recall that $\phi < n$).

We can now determine the equilibrium cut-off level for sender types, $\hat{\theta}$, defined by $j\left(\hat{\theta}\right) = 0$, which we will then use to determine the equilibrium n that we have so far held constant. Since $j\left(\hat{\theta}\right) = 0$, we have

$$\pi\left(\hat{\theta}\right) = \frac{\gamma}{-\ln\left(1 - \phi/n\right)}. \tag{23}$$

The relation above indicates that $\hat{\theta}$ and n are positively related so that a higher threshold firm type, $\hat{\theta}$, is, again, associated to a higher number of messages sent. We can rewrite (23) as

$$n = \frac{\phi}{1 - \exp\left(-\gamma/\pi\left(\hat{\theta}\right)\right)}, \tag{24}$$

which is clearly (and reassuringly) larger than ϕ, as indeed is coherent with our maintained hypothesis that $\phi/n < 1$.

The relation (23) also implies we can rewrite (22) as

$$j\left(\theta\right) = \frac{\pi\left(\hat{\theta}\right)}{\gamma} \ln\left(\frac{\pi\left(\theta\right)}{\pi\left(\hat{\theta}\right)}\right). \tag{25}$$

This is an increasing function of θ, so that more profitable firm types send more messages.

Since $n = M\int_{\hat{\theta}}^{1} j\left(\theta\right) d\theta$, substituting for $j\left(\theta\right)$ from (25) yields

$$n = \frac{M\pi\left(\hat{\theta}\right)}{\gamma} \int_{\hat{\theta}}^{1} \ln\left(\frac{\pi\left(\theta\right)}{\pi\left(\hat{\theta}\right)}\right) d\theta.$$

Hence, we can write this as an implicit function of $\hat{\theta}$ by substituting out for n from (24) to yield

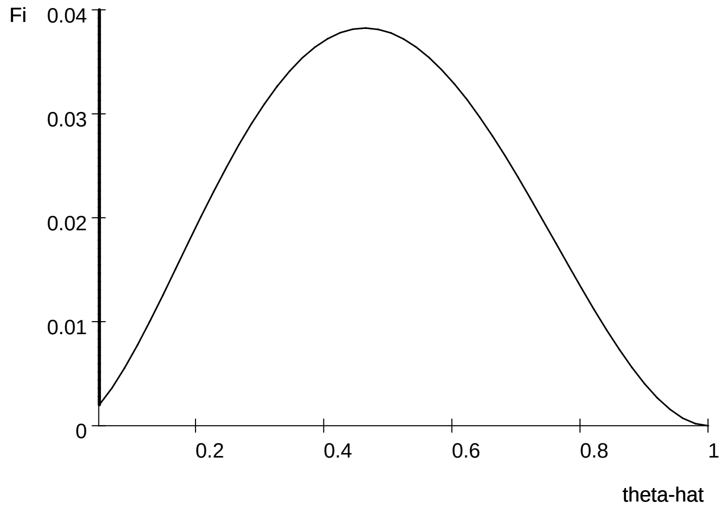
$$\phi = \frac{1 - \exp\left(-\gamma/\pi\left(\hat{\theta}\right)\right)}{\left[-\gamma/\pi\left(\hat{\theta}\right)\right]} M \int_{\hat{\theta}}^{1} \ln\left(\frac{\pi\left(\hat{\theta}\right)}{\pi\left(\theta\right)}\right) d\theta. \tag{26}$$

which is the analogue for the continuous case to the function $\Phi\left(\theta_1\right)$, as given from our six-step procedure in the text.

When $\pi\left(\theta\right)$ is linear in θ (so $\pi\left(\theta\right) = \bar{\pi}\theta$), (26) becomes

$$\frac{\phi}{M} = \frac{-\bar{\pi}\hat{\theta}}{\gamma}\left[1 - \exp\left(-\gamma/\bar{\pi}\hat{\theta}\right)\right]\left[\hat{\theta} - \hat{\theta}\ln\hat{\theta} - 1\right].$$

This case is illustrated in Figure 6 for $\bar{\pi}/\gamma = 2$ and $M = 1$, where we have begun the graph at $\hat{\theta} = 0.05$ to reflect the vertical segment of ϕ at this value which would correspond to the uncongested equilibrium with one message per sender.



Plot of $-20\hat{\theta}\left(1-\exp\left(-1/20\hat{\theta}\right)\right)\left(\hat{\theta}-\hat{\theta}\ln\hat{\theta}-1\right)$ as a function of $\hat{\theta}$.

It can readily be shown that the (normalized) functions $j(\theta)$ for two different equilibria cross only once, analogous to the property in the main text for the continuous approximation we use there.

7.2 Lemma on optimal densities (continuous messages model)

We here give a property of comparing equilibrium message densities to optimal densities under the constraint that the same number of messages be sent in total.

Lemma 9 *At any equilibrium, let the equilibrium number of messages sent, n^e, with corresponding lowest sender type, $\hat{\theta}^e$, and equilibrium density $j^e(\theta)$ (with support $\left[\hat{\theta}^e, 1\right]$). Consider the optimal solution (density of messages, $j^o(\theta)$) for the given number of messages sent, n^e, and the given support , $\hat{\theta}^e$. If $\varepsilon_s < \varepsilon_\pi$ for all $\theta \in (\tilde{\theta}, 1]$, then $j^o(\theta) > j^e(\theta)$ for $\theta \in [\ \hat{\theta}^e, \tilde{\theta})$ and $j^o(\theta) < j^e(\theta)$ for $\theta \in (\tilde{\theta}, 1]$, where $\tilde{\theta}$ is the unique solution to $j^o\left(\tilde{\theta}\right) = j^e\left(\tilde{\theta}\right)$ for $\theta \in [\ \hat{\theta}^e, 1]$, if $\varepsilon_s < \varepsilon_\pi$ for all $\theta \in (\tilde{\theta}, 1]$. If $\varepsilon_s = \varepsilon_\pi$ for all θ, then the equilibrium and optimum densities coincide, $j^o(\theta) = j^e(\theta)$.*

Proof: The optimization problem for the constrained case described in the Lemma is

$$\max_{\{j(\theta)\geq 0\}} \int_{\hat{\theta}^e}^{1} [s(\theta) + \pi(\theta)] \left\{ 1 - \left(1 - \frac{\phi}{n}\right)^{j(\theta)} \right\} d\theta$$

subject to

$$M \int_{\hat{\theta}^e}^{1} j(\theta)\, d\theta = n.$$

Set the multiplier for the constraint as λ. Then, as long as $j(\theta) > 0$, the choice of $j(\theta)$ satisfies:

$$-[s(\theta) + \pi(\theta)] \left\{ \left(1 - \frac{\phi}{n}\right)^{j(\theta)} \ln\left(1 - \frac{\phi}{n}\right) \right\} = \lambda M.$$

Then the slope of the locus satisfies

$$\frac{dj(\theta)}{d\theta} = -\frac{[s'(\theta) + \pi'(\theta)] \left(1 - \frac{\phi}{n}\right)}{[s(\theta) + \pi(\theta)] \ln\left(1 - \frac{\phi}{n}\right)}$$

Recall that the equilibrium satisfies

$$-\pi(\theta) \left\{ \left(1 - \frac{\phi}{n}\right)^{j(\theta)} \ln\left(1 - \frac{\phi}{n}\right) \right\} = \gamma.$$

where the LHS is the marginal profitability of an extra message of type θ. Then the slope of this locus is

$$\frac{dj(\theta)}{d\theta} = -\frac{\pi'(\theta) \left(1 - \frac{\phi}{n}\right)}{\pi(\theta) \ln\left(1 - \frac{\phi}{n}\right)}.$$

Consider any crossing of the optimum and equilibrium loci. Then the optimum is flatter than the equilibrium at such a crossing as

$$-\frac{[s'(\theta) + \pi'(\theta)] \left[1 - \frac{\phi}{n}\right]}{[s(\theta) + \pi(\theta)] \ln\left(1 - \frac{\phi}{n}\right)} < -\frac{\pi'(\theta) \left[1 - \frac{\phi}{n}\right]}{\pi(\theta) \ln\left(1 - \frac{\phi}{n}\right)}$$

Simplifying, this condition becomes

$$[s'(\theta) + \pi'(\theta)]\, \pi(\theta) < \pi'(\theta) [s(\theta) + \pi(\theta)],$$

or

$$\varepsilon_s < \varepsilon_\pi,$$

where the ε's are the relevant elasticities. An important special case has both elasticities equal to 1. Then the slopes are the same at any intersection; since they must cross at least once (because the area underneath each curve is n), they must be coincident: the equilibrium and (constrained) optimum coincide. The case of unit elasticities is that of $s(\theta) = \bar{s}\theta$ and $\pi(\theta) = \bar{\pi}\theta$. Another important case has $s(\theta) = \bar{s}$: then $\varepsilon_s = 0$ and the optimum crosses the equilibrium from above.

This analysis is also useful for evaluating consumer benefits alone (the above addresses the full surplus). In particular, the solution that gives the maximal consumer surplus under the n and $\hat{\theta}$ constraints has slope

$$\frac{dj(\theta)}{d\theta} = -\frac{s'(\theta)\left(1-\frac{\phi}{n}\right)}{s(\theta)\ln\left(1-\frac{\phi}{n}\right)}.$$

Note that this is zero for $s(\theta) = \bar{s}$: the optimal density is therefore uniform. This is clearly the case because all options should have an equal likelihood. Comparing this slope to the equilibrium slope, we find that the optimum is flatter than the equilibrium at any intersection as

$$\frac{s'(\theta)}{s(\theta)} < \frac{\pi'(\theta)}{\pi(\theta)}$$

or $\varepsilon_s < \varepsilon_\pi$. In particular, the equilibrium and (constrained) optimum coincide for linear surplus and profit functions. Q.E.D.

Notice that if $\frac{s'(\theta)}{s(\theta)} < \frac{\pi'(\theta)}{\pi(\theta)}$ for all θ, then the optimum locus is flatter at any θ (which then implies a single crossing of the two loci). For example, linear profit in conjunction with concave surplus satisfy the condition: since $s(0) \geq 0$ (by free disposal – a consumer is never forced to buy an option that would entail negative surplus!) then $s'(\theta) < \frac{s(\theta)}{\theta}$. This means that $\varepsilon_s < 1$.

Linear functions for surplus and profit satisfy the condition of equal elasticities (unity for both). If $\varepsilon_s > \varepsilon_\pi$, the ordering of solutions is reversed and the optimum slopes up faster than the equilibrium (because consumers need more weight than firms give them on the high prospects).

Recall finally that this is a constrained optimization problem: IF there were also the option of free disposal of messages, clearly there would be exactly one message sent at the optimum (for given $\hat{\theta}$).

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