# Games Played through Agents [¤]

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June 15, 2000
Previous draft: July 1999

**Abstract**

We introduce a game of complete information with multiple principals and multiple agents. Each agent makes a decision that can a®ect the payo®s of all principals and all agents. Each principal o®ers monetary transfers to each agent conditional on the action taken by the agent. We characterize pure-strategy equilibria and we provide conditions { in terms of game balancedness { for the existence of an equilibrium with an e±cient outcome. Games played through agents display a type of strategic ine±ciency which is absent when either there is a unique principal or there is a unique agent.

## 1 Introduction

We de¯ne a game played through agents (GPTA) as a game where a set of players (the agents) take decisions that a®ect the payo®s of another set of players (the principals) and the principals can, by means of monetary inducements, try to in°uence the decisions of the agents. In other words, a game played though agents is a multi-principal multi-agent game.

The original principal-agent framework { which has one principal and one agent { has been extended in a general way in two directions: (1) Many principals and one agent (Bernheim and Whinston's [4] common agency); and (2) One principal and many agents (Segal's [17] contracting with externalities). The objective of this paper is to consider the general case with many principals and many agents. Multi-principal multi-agent problems arise in political economy, industrial organization, and labor markets:

[¤]We thank Je® Banks, Bruno Jullien, Michel Le Breton, Gene Grossman, David Levine, Jean-François Mertens, Roger Myerson, David Pérez-Castrillo, Michael Peters, Luca Rigotti, Ilya Segal, Chris Shannon, Eric Van Damme, and seminar participants at Caltech, the Catholic University of Milan, ENTER Jamboree 2000, the Francqui Summer School in Political Economy, Insead, Minnesota, Northwestern, Princeton, Stanford, Tilburg, Toronto, UCLA, Wisconsin-Madison, and Yale for useful discussions.

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**Lobbying** A widespread way of modeling interest group politics is through common agency (e.g. Dixit, Grossman, and Helpman [8]). There are many lobbies (principals) and one politician (the agent). However, the assumption of a unique politician is unrealistic because modern democracies are characterized by a multiplicity of public decision-makers. This is true both in terms of organs (division of powers) and in terms of organ members (many organs { such as parlaments { are collegial). It would be important to know how our understanding of the lobbying process is modi¯ed by the presence of multiple policy makers.[1]

**Vertical Restraints** An industry with several ¯rms (sellers) produces goods that are used by another set of players (buyers), who can be ¯nal consumers or intermediate producers. The sellers propose contracts to the buyers. A contract proposed by one seller may be nonlinear and may cover not only the relation between that supplier and the buyer, but also the relation between the buyer ans the other suppliers, such as an exclusivity clause. These vertical restraints are sometimes viewed as anticompetitive. Members of the Chicago School, in particular Bork [7, p. 280{309], have argued that the contractual arrangements that arise in equilibrium are e±cient from a production point of view. Bernheim and Whinston [5] use common agency to show that the equilibrium contract maximizes the joint surplus of the sellers and the buyer. Is this e±ciency result still true when there are multiple buyers?

**Two-Sided Matching with Monetary Transfers** Firms are looking to hire workers (or sport teams are looking to hire players). A ¯rm can hire many workers. The output of each ¯rm depends on what workers it employs, with the possibility of positive or negative externalities between workers. Workers may have preferences about which ¯rm they work for, and of course they care about salary. Each ¯rm makes a salary o®er to each worker, and then each worker chooses a ¯rm. Is the resulting match in any sense e±cient? This model is taken from P¶erez-Castrillo [14]. More about the connection with P¶erez-Castrillo's work will be said in Section 4, after the main theorem.[2]

A GPTA is de¯ned by a set of principals and a set of agents. Each agent must choose an action out of a feasible set of actions (policy choices in the case of lobbying, quantity orders in supply contracts, or object allocations in auctions). Each principal o®ers to each agent a schedule of monetary transfers contingent on the agent choosing a certain action (campaign contributions, supply contracts, or bids). Given the principals's transfer schedules, an agent chooses his action to maximize the sum of transfers he receives from the principals minus the cost of undertaking the action. A principal chooses his transfer schedules to maximize the utility from the agents' actions minus the sum of transfers he makes to agents.

[1] Groseclose and Snyder's [10] is an exception in that they consider multiple policy-makers. In Section 4, we will consider their vote buying model in detail. See also Grossman and Helpman [11] for a model with multiple lobbies and multiple candidates.

[2] Another interesting example of games played through agents is provided by Besley and Seabright [6] in international taxation: national governments (principals) compete to attract international ¯rms (agents) by o®ering subsidies and tax breaks to ¯rms that relocate on their territory. In some practically relevant cases, this game has only ine±cient equilibria

So far, we have been intentionally vague about timing. The simplest structure is a two-stage game, in which ¯rst all principals simultaneously choose their transfer schedules and then all agents observe the schedules and simultaneously choose their actions. We will work with this timing structure for most of the paper. However, some of the examples above present more complicated timing structures, which include elements of sequentiality. For instance, in a lobbying problem in which two di®erent powers are involved (e.g. local and national, it would be a coincidence if these powers made decisions at the same time. In Section 8 we will consider possible sequential variants of the simultaneous game.

Our main focus is e±ciency, which, in line with the other contributions in this area, is de¯ned as surplus maximization. An outcome is e±cient if it maximizes the sum of the payo®s of all agents and all principals. If there is a unique agent, Bernheim and Whinston have shown that there always exist an equilibrium (the truthful equilibrium) that produces an e±cient outcome. If, instead, there is a unique principal, Segal shows that, if a certain type of externalities among agents' payo® functions is absent, then there always exists an e±cient equilibrium. Hence, in both these limit cases, if there are no direct externalities among agents, e±cient equilibria exist.

This e±ciency result is important in the case of lobbying because it means that the outcome of the in°uence process will maximize the sum of the payo®s of all the players involved in the game, agent and principals. In many models this allows to ¯nd the outcome of lobbying even if we have di±culty ¯nding the equilibrium campaign contributions. It is also important in supply contracts because it gives support to Judge Bork's Thesis.

However, it turns out that, even when there are no direct externalities among agents, a multi-principal multi-agent game need not have an e±cient equilibrium. The presence of multiple players on both sides creates a strategic externality that makes it impossible to achieve the e±cient outcome. The main result of this paper is to provide a general necessary and su±cient condition for the existence of an e±cient equilibrium. This condition relates to the cooperative concept of balancedness, which we extend { with some important di®erences { to our game.

In the present context, balancedness has a noncooperative interpretation in terms of weighted deviations from the equilibrium outcome and sheds light on the nature of the strategic interaction between principals and agents. The balancedness of a game can be checked in a straightforward way by computing the value of a linear program. Balancedness can also be used to show that, if the principal's payo® functions are continuous and convex, then there always exists an e±cient equilibrium. This last result is used to show to ¯nd simple su±cient conditions under which Bork's Thesis is correct.

The connection between GPTA's and cooperative concepts leads to a noncooperative foundation of the core. Every cooperative game with transferable utility (TUG) can be put in correspondence with a GPTA, such that the TUG has a nonempty core if and only if the corresponding GPTA has a pure-strategy equilibrium. However, a generic GPTA cannot be rewritten as a TUG.

The organization of the paper is as follows. As the existence of ine±ciency in our model does not depend on externalities among agents, we develop our core argument under the assumption that each agent cares only about the action he takes and the money he gets (results on the case with direct externalities are reported in appendix).

We begin with the formal presentation of the game in Section 2. In Section 3 we give a characterization of pure-strategy equilibria that we will use in the rest of the paper. In Section 4, we focus on a simplified version of the game, in which each agent has only two possible actions. This simplification avoids issues of coordination among principals. The main result is a necessary and sufficient condition for the existence of an efficient equilibrium, which we then discuss in relation of to the literature. In Section 5 we allow agents to have more than two actions. To deal with coordination problems, we introduce and study weakly truthful equilibria, which are an extension of Bernheim and Whinston's truthful equilibrium to games with many agents. We give necessary and sufficient conditions for their existence, again in terms of balancedness. In Section 6, we use these conditions to show that in a convex environment there always exists an efficient equilibrium, which in turn leads to a formal statement of Judge Bork's claim on the efficiency of vertical restraints. In Section 7 we show that a mixed-strategy equilibrium must always exist. In Section 8, we probe the robustness of our results against alternative timing structures, in which either principals move sequentially (principal sequentiality) or agents move sequentially (agent sequentiality). Section 9 allows principals to offer outcome-contingent contracts, that is transfers conditional not only on what the agent who is offered the transfer does, but also on what all other agents do. It is shown that this extension does not restore efficiency. Section 10 concludes by examining the implications of our results for the three examples of games played through agents discussed above.

For completeness, we also report results on the case where agents have direct externalities (Appendix A) and more detailed results on the principal-sequential version (Appendix B). Most proofs are collected in Appendix C.

## 2 Games Played through Agents

There is a set $M$ of principals and a set $N$ of agents. Let $m$ denote the typical element of $M$ and $n$ the typical element of $N$. We emphasize that there is no natural relation between any of the principals and any of the agents. The game takes place in two stages: first the principals move simultaneously, then the agents move simultaneously.[3]

Each agent has a finite pure set of actions $S_n$. Let $S \equiv \prod_{n \in N} S_n$ and $H$ the disjoint union of the sets $S_n$ over $n \in N$. The typical element of $S_n$ is denoted by $s_n$, the typical element of $S$ is tuple $s = (s_1; \ldots; s_n)$. Each element of of $H$ is naturally associated to the pair $(n; s_n)$ specifying the agent and the action of that agent. We write in the following an element of $H$ as a pair $(n; s_n)$. Each principal chooses a vector of nonnegative transfers $t^m \in \Re_+^H$ which specifies the transfer from her to each agent for each action of that agent. Thus, $t_n^m(s_n)$ is the tranfer of principal $m$ to agent $n$ conditional on agent $n$ choosing action $s_n$. Agent $n$ receives money only for the action that he actually chooses, but he may receive money from more than one principal.

We have assumed that the transfer from a principal to an agent can only be contingent on the action chosen by that agent. The transfer could depend also on the actions chosen by other agents, in which case we would write $t_n^m(s)$ instead of $t_n^m(s_n)$. Section 9 will examine this type of transfers, which we call outcome-contingent. The main result is that

[3]Sequential variations of the game are considered in Section 8.

allowing for outcome-contingent transfers does not solve the efficiency problem.[4]

Each agent cares about what action he chooses and how much money he gets. Agent n's payoff if principals offer t and he selects $s_n$ is $F_n(s_n) + \sum_{m \in M} t_n^m(s_n)$. From Segal [17] we know that in the one-principal case, if agents care about the actions taken by other agents, then there need not be an efficient equilibrium. Thus, for our purpose it is interesting to restrict attention to the utility function of agent n to include his action but not actions taken by other agents. However, the more general case in which $F_n(s_n)$ is substituted with $F_n(s)$ is considered in Appendix 11. The term $F_n(s_n)$ can be interpreted as cost of effort in the principal-agent tradition.

Principals care both about money and the actions that agents choose. Let $G^m(s)$ be the gross payoff to Principal m if action s is chosen by the agents. The net payoff of principals is assumed to be separable in gross payoff and money. The net payoff to Principal m if she offers transfers $\{t_n^m(s_n)\}_{(n;s_n) \in H}$ and agents choose $\hat{s}$ is $G^m(\hat{s}) - \sum_{n \in N} t_n^m(\hat{s}^n)$.[5]

Throughout the paper, we adopt the convention that superscripts denote principals, subscripts denote agents, while arguments are reserved for actions or outcomes.

The extensive form game is as follows. First, each principal chooses her vector of transfers to the agents simultaneously and noncooperatively. Second, the vectors of all principals are publicly announced to agents, who then choose their actions. Although it is not crucial, we assume that each agent also observes offers made to other agents.

We focus here on pure strategies (Mixed strategies are considered in Section 7). The strategy set of Principal m is the subset $T^m \equiv \Re^{+H}$. A pure strategy for m is simply an element of $T^m$. Let $T \equiv \times_{m \in M} T^m$. The action set for agent n is $S_n$. A pure strategy for Agent n is $\sigma_n : T \to S_n$. A pure-strategy equilibrium is a subgame-perfect equilibrium of the two-stage game in which each agent and each principals uses a pure-strategy:

**Definition 1** A pure strategy equilibrium of a GPTA is a pair $(\hat{t}; \hat{\sigma})$, where $\hat{t} = (\hat{t}_n^m(s_n))_{m \in M; n \in N; s_n \in S_n}$ and $\hat{\sigma} = (\hat{\sigma}_n)_{n \in N}$, in which:

(i) For every $n \in N$, and every $t \in T$,

$$\hat{\sigma}_n(t) \in \operatorname{argmax}_{s_n \in S_n} F_n(s_n) + \sum_{m \in M} t_n^m(s_n)$$

(ii) For every $m \in M$, given $(\hat{t}^j)_{j \neq m}$, $\hat{t}^m$ solves:

$$\max_{t^m} G^m(\hat{\sigma}_n(t^m; \hat{t}^{-m})) - \sum_{n \in N} t_n^m(\hat{\sigma}_n(t^m; \hat{t}^{-m}))$$

We now define efficiency. An action is efficient if it maximizes the sum of the net payoffs of all players (agents and principals). Transfers can be neglected, and the definition of efficiency is:

[4] One might also allow principals to use more complex mechanisms (Epstein and Peters [9]). However, this seemingly difficult problem is left for future research.

[5] The separability assumption for both principals and agents does not appear to be crucial to the results presented here, as it is not crucial to the results obtained in common agency (Dixit, Grossman, and Helpman [8]).

**Definition 2** Action $s^*$ is efficient if

$$\sum_{n \in N} F_n(s_n^*) + \sum_{m \in M} G^m(s^*) \geq \sum_{n \in N} F_n(s_n) + \sum_{m \in M} G^m(s) \qquad (1)$$

for every $s \in S$.

The outcome of an equilibrium is the action profile chosen by agent in that equilibrium. We will sometimes say that an "equilibrium is efficient," meaning that the outcome of that equilibrium is efficient.

If we take our framework and let $M = \{1\}$, we have Segal [17] with no direct externalities, and we know from his Proposition 1 that the game always has an efficient equilibrium. If instead we take $N = \{1\}$, we get Bernheim and Whinston [4] and, again, we know that the game always has an efficient equilibrium. The question we ask here is whether efficiency also holds for a generic $M$ and $N$.

## 3 A First Characterization of Pure Strategy Equilibria

A pure strategy equilibrium is characterized by three conditions, which are formally reported in Theorem 1 below. They are derived using the idea, common in principal-agent problems, that we may think of principals as choosing the action of the agents, provided they give the appropriate incentive to the agents. The conditions are:

1. Each agent chooses an action that maximizes his payoff, given the transfers of the principals. This is the condition (AM) (Agent Maximization) below.

2. Given the transfers of the other principals, Principal $m$ can induce agents to choose any particular action provided that she puts high enough transfers on that action. The minimum cost for $m$ to convince agent $n$ to move from $\hat{s}_n$ to $s_n$ is $F_n(s_n) + \sum_{j \neq m} \hat{t}_n^j(s_n) - F_n(\hat{s}_n) - \sum_{j \neq m} \hat{t}_n^j(\hat{s}_n)$. If $\hat{s}$ is an equilibrium, then the cost of a deviation must be greater than the benefit of a deviation for each $m$ and each $s$, which is what Condition (IC) (Incentive Compatibility) requires.

3. Each principal sets his transfers so that the cost of implementing $\hat{s}$ is minimal. There cannot be a way in which principal $m$ reduces the equilibrium transfers for $\hat{s}$ without deviating from $\hat{s}$. This is condition (CM) (Cost Minimization).[6]

Formally, this is the characterization.

**Theorem 1** A pair $(\hat{t}, \hat{s})$ of transfers and action profiles is a pure strategy equilibrium outcome if and only if the following conditions are satisfied:

[6] Note that what we call (AM) is what is usually called incentive-compatibility in principal-agent problems. However, it is useful here to use the term incentive compatibility for the principals' choices. While the agent maximization problem is defined by one condition (AM), the principal maximization problem is defined by two conditions (IC) and (CM). Then it is useful to distinguish between principal incentive compatibility (which is across actions) and principal cost minimization (which is for the equilibrium action).

(AM) for every $n \in N$, $s_n \in S_n$,

$$F_n(\hat{s}_n) + \sum_{m \in M} \hat{t}_n^m(\hat{s}_n) \geq F_n(s_n) + \sum_{m \in M} \hat{t}_n^m(s_n);$$

(IC) for every $m \in M$, $s \in S$,

$$G^m(\hat{s}) + \sum_{n \in N} F_n(\hat{s}_n) + \sum_{n \in N} \sum_{j \neq m} \hat{t}_n^j(\hat{s}_n) \geq G^m(s) + \sum_{n \in N} F_n(s_n) + \sum_{n \in N} \sum_{j \neq m} \hat{t}_n^j(s_n);$$

(CM) for every $m \in M$, $n \in N$,

$$F_n(\hat{s}_n) + \sum_{m \in M} \hat{t}_n^m(\hat{s}_n) = \max_{s_n \in S_n} \left( F_n(s_n) + \sum_{j \neq m} \hat{t}_n^j(s_n) \right):$$

Some of the properties of the equilibrium are worth pointing out explicitly:

**Corollary 1** for all $n$, either $\sum_{m \in M} t_n^m(\hat{s}_n) = 0$, or there is an $\bar{s}_n \in S_n \setminus \{\hat{s}_n\}$ such that:

$$F_n(\hat{s}_n) + \sum_{m \in M} \hat{t}_n^m(\hat{s}_n) = F_n(\bar{s}_n) + \sum_{m \in M} t_n^m(\bar{s}_n): \qquad (2)$$

For every agent, either the agent gets no money for the equilibrium action, or he must be exactly indifferent between choosing the equilibrium action and choosing another action. Otherwise, some principal could reduce the transfers she offers for the equilibrium action.

Moreover, it is immediate from Corollary 1 and (CM), that, for every $n$ and every $m$, there exists an action $a(m; n)$ (which could be $\hat{s}_n$) such that

$$\text{(i) } F_n(a) + \sum_{j \in M} t_n^j(a) = F_n(\hat{s}_n) + \sum_{j \in M} t_n^j(\hat{s}_n); \text{ and (ii) } t_n^m(a) = 0: \qquad (3)$$

Given a principal and an agent, there always exists an action in the set of actions that maximize the payoff of the agent for which that particular principal offer a zero transfer. This action could be the equilibrium action or some other competing alternative.

## 4 Agents with Two Actions

In this section we introduce the main results of the paper in a simplified environment in which each agent has only two actions and he does not care directly about the action he chooses. We proceed as follows: Subsection 4.1 restates the characterization theorem in this simplifies environment. Subsection 4.2 provides four examples. Subsection 4.2 states the main theorem: a necessary and sufficient condition for the existence of an efficient equilibrium. Subsection 4.3 discusses the condition.

## 4.1 Characterization

We denote by $s'_n$ the action of $n$ different from $\hat{s}_n$. The fact that agent $n$ is indifferent between action is expressed as $F_n(\hat{s}_n) = F_n(s'_n) = 0$. The following is an immediate restatement of Theorem 1, combined with corollary 1;in this simplified environment:

**Proposition 1** For every $n$, suppose that $\#S_n = 2$ and that $F_n(\hat{s}_n) = F_n(s'_n) = 0$. The pair $(\hat{t}, \hat{s})$ is a pure strategy equilibrium outcome if and only if

(AM) For every $n \in N$, $s \in S$

$$\sum_{m \in M} \hat{t}_n^m(\hat{s}_n) = \sum_{m \in M} \hat{t}_n^m(s'_n);$$

(IC) For every $m \in M$, $s \in S$:

$$G^m(\hat{s}) + \sum_{n \in N} \sum_{j \neq m} \hat{t}_n^j(\hat{s}_n) \geq G^m(s) + \sum_{n \in N} \sum_{j \neq m} \hat{t}_n^j(s_n);$$

(CM) For every $m \in M$, $n \in N$,

$$\text{if } \hat{t}_n^m(\hat{s}_n) > 0 \text{ then } \hat{t}_n^m(s'_n) = 0:$$

With two actions per agent, no principal can make at equilibrium a strictly positive transfer on more than one of the two actions, for each agent, since this would immediately violate the condition (CM). Also by (AM) and the fact that agents do not care about actions directly, the sum of transfers for one action is exactly equal to the sum of transfers for the other action.

A couple of observations are straightforward from Proposition 1. By summing (AM) over $n$ and subtracting it from (IC) applied to $s = (s'_n; \hat{s}_{-n})$, we have that for all $n$

$$G^m(\hat{s}) - G^m(s'_n; \hat{s}_{-n}) \geq \hat{t}_n^m(\hat{s}_n) - \hat{t}_n^m(s'_n)$$

which, combined with (CM) implies that for any $m$ and any $n$

$$\begin{array}{rl} \text{either} & \hat{t}_n^m(\hat{s}_n) = 0 \text{ and } \hat{t}_n^m(s'_n) \geq G^m(s'_n; \hat{s}_{-n}) - G^m(\hat{s}) \\ \text{or} & 0 < \hat{t}_n^m(\hat{s}_n) \leq G^m(\hat{s}) - G^m(s'_n; \hat{s}_{-n}) \text{ and } \hat{t}_n^m(s'_n) = 0: \end{array} \quad (4)$$

For an example of the implications of this remark, see the argument immediately below 5. Moreover, in the two actions case, the action outcome of a pure strategy equilibrium is efficient:

**Proposition 2** If $\#S_n = 2$ for every $n \in N$, then pure strategy equilibria are efficient.

The assumption that $\#S_n = 2$ is essential. As we shall see in Section 5, with more than three actions a pure strategy equilibrium need not be efficient. Instead, the assumption that agents do not care about actions is not essential.

## 4.2 Examples

We consider a few examples with $M = N = \{1, 2\}$. We denote the agents' actions as $S_1 = \{T, B\}$ and $S_2 = \{L, R\}$. We adopt the convention of presenting the payoff matrix in the form:

| | L | R |
|---|---|---|
| T | $G^1(TL); G^2(TL)$ | $G^1(TR); G^2(TR)$ |
| B | $G^1(BL); G^2(BL)$ | $G^1(BR); G^2(BR)$ |

It is important to keep in mind that this is not the usual payoff matrix. The actions refer to agents while the payoffs refer to principals. As in the previous section, agents have no direct interest in the action they choose. Also, these are gross payoffs. The net payoffs will be given by the gross payoffs minus the transfers. The transfer vector $t^m$ is written as $(t_1^m(T); t_1^m(B); t_2^m(L); t_1^m(R))$.

**Prisoner's Dilemma** The payoffs of the principals are:

| | L | R |
|---|---|---|
| T | $x; x$ | $z; y$ |
| B | $y; z$ | $0; 0$ |

with $y > x > 0 > z$ and $2x > y + z$. The efficient action is unique: $(T; L)$. Hence, by Proposition 2, if a pure-strategy equilibrium exists, it must have $(T; L)$ as outcome. By (CM), the first principal can make at equilibrium a positive transfer either on $T$ or on $B$. But $t_1^1(T) > 0$ would require by equation 4 that $G^1(TL) - G^1(BL) > 0$, which is not the case. Hence he does not pay for the action $T$. A similar argument shows that the second does not pay for $L$. By (AM), the payment on each action from the two principals must be the same; so the equilibrium transfers are pairs of the form:

$$t^1 = (0; a; b; 0); t^2 = (a; 0; 0; b): \tag{5}$$

The (IC) condition for the first principal is $x + a \geq \max\{z + a + b; y; b\}$, and $x + b \geq \max\{z + a + b; y; a\}$ for the second. So the set of pure strategy equilibria is given by any transfer with $(a; b)$ such that $a; b \in [y - x; x - z]$ and $x \geq b - a \geq -x$. In particular, there exists a minimal transfer equilibrium in which $a = b = y - x$. The agents choose $T$, respectively $L$, whenever indifferent. The rent of each agent is at least the difference between the best outcome and the "cooperate" outcome, $y - x$, and at most the difference between "cooperate" and the bad outcome, $x - z$.

**Coordination Game** The payoffs of the principals are:

| | L | R |
|---|---|---|
| T | $x_1; y_1$ | $0; 0$ |
| B | $0; 0$ | $x_2; y_2$ |

with

$$x_1 + y_1 > x_2 + y_2; y_1 \leq y_2; x_i; y_i \geq 0; i = 1; 2$$

Again there is a unique efficient outcome, $(T, L)$, hence a unique equilibrium outcome in pure strategy, if any. From Proposition 1 it is easy to see that there exists an equilibrium if there are transfers $t^1 = (a, 0, b, 0)$, $t^2 = (0, a, 0, b)$ that satisfy

$$x_1 - x_2 \geq a + b \geq y_2 - y_1, \tag{6}$$

Clearly, (6) is satisfied for the parameters under consideration and an equilibrium with outcome $(T, L)$ always exists. The combined rent of the two agents is at least $y_2 - y_1$. The symmetric equilibrium, with transfers $t^1 = (0, a, 0, b)$ and $t^2 = (a, 0, b, 0)$ exists if

$$y_1 - y_2 \geq a + b \geq x_2 - x_1. \tag{7}$$

It has combined transfers at least as large as $x_2 - x_1$.

But there are also equilibria with transfers $t^1 = (0, a, b, 0)$ and $t^2 = (a, 0, 0, b)$ if

$$x_1 - x_2 \geq b - a \geq y_2 - y_1, x_1 \geq b, y_1 \geq a. \tag{8}$$

In these equilibria the agents may extract the full rent from the principals.

In the extreme "pure", coordination game with $x_1 = y_1 > x_2 = y_2$, zero transfers from both principals, for each agent and each action is an equilibrium. But also the vector of transfers $t^1 = (0, x_1, x_1, 0)$ and $t^2 = (x_1, 0, 0, x_1)$ is an equilibrium. Each principal is giving mis-matched transfers: the first is paying the first agent to do $B$, but the second to do $L$. This equilibrium leaves no surplus to the principals.

The prisoners' dilemma and the coordination game, when played through agents, have a unique pure-strategy equilibrium, and this equilibrium is efficient. The following examples instead show that there are games in which a pure-strategy equilibrium does not exist and all other equilibria are inefficient.

**Matching Pennies** Let

| | L | R |
|---|---|---|
| T | $x_1, 0$ | $0, y_1$ |
| B | $0, y_2$ | $x_2, 0$ |

with $x_1$ the largest number. The only possible pure strategy equilibrium outcome is $(T, L)$, with transfers $t^1 = (a, 0, b, 0)$ and $t^2 = (0, a, 0, b)$. The condition IC for the first principals is $x_1 \geq \max\{a, b, x_2 + a + b\}$, for the second $a + b \geq y_1 + a, y_2 + b$, which are equivalent to $x_1 - x_2 \geq a + b \geq y_1 + y_2$, so a pure strategy equilibrium exists if and only if $x_1 - x_2 \geq y_1 + y_2$. The maximum total rent of the agents is the difference between the payoffs of the first principal, and the minimum total rent is the sum of the payoffs of the second principal.

In particular a pure strategy equilibrium does not exist for the "true" matching pennies, with all the numbers equal to 1.

**Opposite Interests Game** In this game the payoff matrix is

| | L | R |
|---|---|---|
| T | $3, 0$ | $0, x$ |
| B | $0, x$ | $0, x$ |

with $1.5 < x < 3$. By Proposition 2, the only possible pure strategy equilibrium outcome is $(T, L)$, and the second principal can only pay for the action B and R. So the possible transfers are $t^1 = (a, 0, b, 0)$ and $t^2 = (0, a, 0, b)$. The (IC) for the two principals are, respectively:

$$3 \geq \max\{a, b, a + b\};$$

$$\min\{a, b, a + b\} \geq x.$$

Together, they imply $3 \geq a + b$ and $x \leq \min\{a, b\}$, which cannot be satisfied if $x > 1.5$. No pure strategy equilibrium exists and all other equilibria are necessarily inefficient because they involve outcomes different from $(T, L)$ with positive probability.

The opposite interest game can be interpreted as an example of each of the three applications of GPTA's proposed in the Introduction. It can be seen as a lobbying problem where Principal 1 is a lobby who wants to change the status quo and Principal 2 wants to keep things as they are. In order to change the status quo, Principal 1 needs the unanimous approval from two governmental bodies, Agent 1 and Agent 2. The efficient outcome is to change the status quo. However, Principal 2 enjoys a strategic advantage because he only needs to convince one of the two agents to say no.

With some re-working, the Opposite Interest Game can also be interpreted as a very basic vertical contracting problem with two sellers (principals) and two buyers (agents). Each buyer needs exactly one unit of the input good produced by the sellers. The total cost functions of the two sellers are as follows:

| $q$ | 0 | 1 | 2 |
|---|---|---|---|
| $C^1$ | 0 | 3 | 3 |
| $C^2$ | 0 | $3 - x$ | $6 - x$ |

Seller 1 has economies of scale while Seller 2 has diseconomies. The efficient allocation would be that 1 produces two units and 2 produces nothing. Let T represent Buyer 1 buying his unit from Seller 1 and let L represent Buyer 2 buying his unit from Seller 2. B and R are the opposite actions. Suppose that there is a 'fixed' price of 3 per unit but sellers can offer discounts (this is a quick way to overcome the non-negativity constraint – the whole analysis of this paper can be redone with a non-positivity constraint or with other constraints). For instance, $t^1_2(L)$ is the discount over the fixed price of 3 that Principal 1 offers to Agent 1 if he buys from her. Then, it is easy to check that this supply contract problem is exactly equivalent to the Opposite Interest Game examined above and, therefore, has no efficient equilibrium. In order to achieve efficiency, Principal 1 should sell to both buyers but Principal 2 can easily undercut her on one of the two buyers. The noncontractible externality here is that, if Principal 2 sells to Buyer 2, there is an increase in the cost of production for the good that Principal 1 is still selling to Agent 1.

One can also view the Opposite Interest Game as a two-sided matching problem with two firms and two workers. Firm 1 displays strong positive complementarities between works, while firm 2 displays strong negative complementarities.[7]

**Voting Game** Our last example has more than 2 agents and is related to Groseclose and Snyder [10]. There are two principals, $M = \{1, 2\}$, and an odd number $N = 2K + 1$ of agents. Each agent may vote for one of two alternatives, also labelled 1 and 2 and he may not abstain. The alternative with the larger number of votes is chosen. The payoff of the principal 1 is $x \geq 1$ if the alternative 1 is chosen, and 0 if 2 is chosen. The payoff of Principal 2 is 1 if 2 is chosen and 0 otherwise. Thus, all action profiles such that $\#\{n \in N | s_n = 1\} \geq K + 1$ are efficient.

This game has no equilibrium in which alternative 1 is chosen for sure, and hence it has only inefficient equilibria. To see this, suppose that an equilibrium exists, where alternative 1 is chosen for sure. In this equilibrium, Principal 2 must be paying no money to agents. If it were not so, Principal 2 would get a negative payoff while she can always ensure a zero payoff by offering zero to all agents. There are two cases: (i) Principal 1 makes a strictly positive offer for certain to all agents; (ii) There is an agent n that receives zero offers from both Principal 1 and Principal 2. In case (i), given any strategy of Principal 2, Principal 1 can still guarantee herself Alternative 1 but save money by making a zero offer to one of the agents. In case (ii) Principal 1 could offer a zero transfer to one of the agents she is currently offering a strictly positive transfer and make an infinitesimal transfer to the agent who is not receiving anything. This shows that no equilibrium in which Alternative 1 is chosen for sure exists.[8]

## Existence of pure strategy equilibria: Necessary and sufficient condition

In this section we provide a necessary and sufficient condition for the existence of a pure-strategy equilibrium. Our interest in the set of pure-strategy equilibria is due to its essential equivalence with the set of efficient equilibria. We know that pure-strategy equilibria are efficient (Proposition 2). Moreover, if some principals were using mixed strategies on transfers, necessarily some agents would be choosing both actions with positive probability. Hence, in a generic GPTA – which has a unique efficient outcome – a mixed strategy equilibrium selects an inefficient outcome with positive probability.

Before introducing the formal analysis, we motivate our definitions by considering a game with two principals and two agents, with $S_n = \{1, 2\}$ for both agents.

As an illustration consider the case of a game with two principals and two agents, with $S_n = \{1, 2\}$ for both agents, and $\hat{s} = (1, 1)$. One of the necessary conditions (??) is, for instance:

$$G^1(11) - G^1(22) + G^2(11) - G^2(12) + G^2(11) - G^2(21) \geq 0:$$

[7] Having two auctions simultaneously is not realistic. In Subsection 8.2 we will look into the agent-sequential version of the Opposite Interest Game and interpret it as sequence of two auctions.

[8] Groseclose and Snyder [10] present the game in a sequential form. First Principal 1 makes offers. Then, Principal 2 observes the offers made by 1 and makes her offers. They show that a principal may want to buy a supermajority, that is, make a positive offer to strictly more than $K + 1$ agents.

We rewrite this condition in the somewhat cumbersome way:

$$\sum_{s\in S} w^1(s)(G^1(\hat{s}) - G^1(s)) + \sum_{s\in S} w^2(s)(G^2(\hat{s}) - G^2(s)) \geq 0. \quad (9)$$

where

$$w^1(22) = w^2(12) = w^2(21) = 1 \quad (10)$$
$$w^1(11) = w^1(12) = w^1(21) = w^2(11) = w^2(22) = 0.$$

The idea is that the w's are weights that principals put on possible deviations: $w^m(s)$ is the weight Principal m puts on a deviation from $\hat{s}$ to s. The weights in (10) satisfy

$$w^1(12) + w^1(22) = w^2(12) + w^2(22); \quad (11)$$
$$w^1(21) + w^1(22) = w^2(21) + w^2(22). \quad (12)$$

Condition (11) says that the sum of weights on all the deviations which involve the participation of Agent 1 is the same for the two principals. Condition (12) is the same condition for Agent 2. We make this into a general definition:

**Definition 3** If agents have only two actions, $(w^m(s))_{m\in M; s\in S}$ is said to be a collection of balanced weights if $w^m(s) \geq 0$ for every m and s, and

$$\text{for every } m \in M; n \in N \quad \sum_{\{s: s_n \neq \hat{s}_n\}} w^m(s) = \sum_{\{s: s_n \neq \hat{s}_n\}} w^1(s). \quad (13)$$

This definition generalizes (11) and (12), since $\{s : s_n \neq \hat{s}_n\}$ is the set of possible deviations that involve the participation of Agent n. The sum of weights over this set must be constant across principals.

Now, reconsider (9). It asks that the sum of benefits from a deviation from 11 to 22, weighted according to a particular vector of balanced weight, be nonnegative. We generalize the condition as follows:

**Definition 4** A GPTA is balanced if and only if for every vector of balanced weights $(w^m(s))_{m\in M; s\in S}$ we have:

$$\sum_{m\in M} \sum_{s\in S} w^m(s)(G^m(\hat{s}) - G^m(s)) \geq 0. \quad (14)$$

Our definition of balancedness is different from the definition of balancedness used in cooperative game theory (e.g. Scarf [16]) because of the distinction in our game between principals and agents. However, the interpretation of our condition follows the lines of the traditional interpretation of balancedness in terms of "time" devoted to coalitions. To see this interpretation, note that the weights w can be re-scaled without loss of generality so that $\sum_{n\in N} \sum_{\{s: s_n \neq \hat{s}^n\}} w^m(s) = 1$ for every m. This means that each principal is allocated a unitary amount of time to "convince" agents to choose or not to choose a particular deviation. The first step consists in allocating time across agents. This allocation will be the same for all principals: $\sum_{\{s: s_n \neq \hat{s}^n\}} w^m(s)$. Each principal is given a time $\sum_{\{s: s_n \neq \hat{s}_n\}} w^m(s)$ to convince agent n to deviate or not to deviate from $\hat{s}_n$ to $s_n'$. The second step is to decide the form of these deviations, which can involve any number of agents. For instance, Principal m may talk to Agent n about deviating by himself or may try to convince agent n and another agent to deviate together. The third step consists in evaluating the "goodness" of a deviation, which is given by $w^m(s)(G^m(s) - G^m(\hat{s}))$, that is, the time spent to convince the agents involeved in the coalition multipled by the benefit of the deviation for principal m. A deviation will be successful if $\sum_{m\in M}\sum_{s\in S} w^m(s)(G^m(s) - G^m(\hat{s})) > 0$. Thus, a game is balanced if, for all time allocations across agents and possible "deviation arguments", there is no successful deviation.[9]

By summing (13) over n, we obtain that if the weight corresponding to each deviation is multiplied by the number of agents that must deviate to realize that deviation, then the sum is constant:

**Proposition 3** If a vector $(w^m(s))_{m\in M; s\in S}$ is a vector of balanced weights, then:

$$\text{for every } m \in M; \quad \sum_{s\in S} w^m(s)\#\{n : s_n \neq \hat{s}_n\} = \sum_{s\in S} w^1(s)\#\{n : s_n \neq \hat{s}_n\}.$$

This is in agreement with the previous interpretation. The 'cost' of a deviation is proportional to the number of agents that must be convinced. The total cost must equal the endowment. Hence, deviations that involve a small numbers of agents are cheaper than deviations with many agents. In the Opposite Interest Game, Principal 2 had two cheap deviations and that made it hard for Principal 1 to defend the efficient outcome.

We are now ready to state our main result:

**Theorem 2** A GPTA where agents have two actions has a pure strategy equilibrium if and only if it is balanced.

Proof: From proposition 1, a pure strategy equilibrium exists if and only if the three conditions of that proposition hold. If we denote

$$d_n^j \equiv t_n^j(\hat{s}_n) - t_n^j(s_n')$$

(AM) and (IC) may be rewritten as

$$\sum_{\{j: j\neq k\}}\sum_{\{n: s_n\neq \hat{s}_n\}} d_n^j \geq G^m(s) - G^m(\hat{s}) \qquad \forall s\in S; \forall k \in M; \tag{15}$$

$$\sum_{j\in M} d_n^j = 0 \qquad \forall n \in N. \tag{16}$$

The system (15) and (16) is a system of linear inequalities in the $M \times N$ variables $d_n^j$. There are $M \times S$ inequalities of the type in (15) and $N$ inequalities of the type (16).

We can find a d that solves (15) and (16) if and only if we can find a t that solves (AM), (IC), and (CM). The "if" part is by definition. The "only if" part can be seen as follows. Suppose we find a d that solves (15) and (16). Let $t_n^j(\hat{s}_n) = \max(0; -d_n^j)$ and $t_n^j(\hat{s}_n') = \max(0; d_n^j)$. The resulting t satisfies (AM), (IC), and (CM).

The following result is well known, and reported here only for convenience:[10]

[9] As the simple example of balanced weights (10) shows, the sum over s of the weights need not be constant: so, in particular, weights cannot be interpreted as probabilities.

[10] See for instance Mangasarian [12].

**Theorem 3 (Farkas)** Exactly one of the following alternatives is true: (a) There exists a solution $x$ to the linear system of (in)equalities given by $Ax \geq a$ and $Bx = b$; or (b) There exist vectors $\mu$ and $\lambda$ such that: (i) $\mu A + \lambda B = 0$; (ii) $\mu \geq 0$; and (iii) $\mu a + \lambda b > 0$.

We now apply Farkas' Lemma to (15) and (16). For $m, j \in M$, $i, n \in N$, $s \in S$, let

$$A_{(ms;jn)} = \begin{cases} 1 \text{ if } j \neq m; s_n \neq \hat{s}_n; \\ 0 \text{ otherwise;} \end{cases}$$

$$B_{(i;jn)} = \begin{cases} 1 \text{ if } j = i; \\ 0 \text{ otherwise.} \end{cases}$$

$$a_{ms} = G^m(s) - G^m(\hat{s}) \quad (17)$$

$$b_i = 0 \quad (18)$$

$$(19)$$

Then, (15) and (16) rewrite as $Ad \geq a$ and $Bd = b$. By Farkas' Lemma a solution $(d^j_n)_{j \in M; n \in N}$ of that system exists if and only if there is no solution $((w^m(s)_{m \in M; s \in S}; (\lambda_i)_{i \in N})$ of the system:

$$\sum_{m \in M; s \in S} w^m(s) A_{(ms;jn)} + \sum_{i \in N} \lambda_i B_{(i;jn)} = 0 \qquad \forall j \in M; \forall n \in N; \quad (20)$$

$$w^m(s) \geq 0 \qquad \forall m \in M; \forall s \in S;$$

and

$$\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(s) - G^m(\hat{s})) > 0. \quad (21)$$

The system (20) may be rewritten as:

$$\text{for every } j \in M; n \in N; \quad \sum_{\{s: s_n \neq \hat{s}_n\}} w^m(s) = -\lambda_n. \quad (22)$$

As this is the only restriction that the variables $\lambda$ are imposing, (22) is true if and only if $w$ is a vector of balanced weights.

Inequality (21) is the negation that the game is balanced for a particular vector of weights. The lack of solution for the system (20) and (22) is equivalent to the requirement that for all balanced weights the inequality

$$\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) \geq 0$$

holds, and this is the statement we had to prove. ■

Theorem 2 is a duality result. Either the system that characterizes pure-strategy equilibria has a solution or it is possible to find a collection of weights that violates the balancedness condition. Thus, in the case of the Opposite Interest Game, where we already know that an efficient equilibrium does not exist, the theorem tells us that there must exist a collection of balanced weights that violates balancedness. Indeed, a successful deviation is the one discussed above in (10). Both principals are allocated a unitary amount of time to be spent half on one agent and half on the other. Principal 1 spends her time to convince her agents (not) to take a joint deviation from $TL$ to $BR$. Principal 2 instead argues for independent deviations: Agent 1 chooses $B$ instead of $T$, and agent 2 chooses $L$ instead of $R$. With this time allocation, the deviation is successful, as shown in (9).

While in the Opposite Interest Game – which has a symmetric structure – we had already given a direct proof that no efficient equilibrium exists, in general direct proofs are not straightforward. Instead, Theorem 2 suggests a simple algorithm to ascertain whether a particular GPTA has an efficient equilibrium. Take $\hat{s}$ to be an efficient outcome (in the nongeneric case when there are many, one must examine each of them). Balancedness corresponds to a minimization program in which the objective function is $\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(\hat{s}) - G^m(s))$ and the control variables are the $w$. The minimization is subject to the constraint that $w$ is balanced. The game is balanced if and only if the value of the objective function is nonnegative. As the objective function and the constraints are linear in the control variables, this is a linear programming problem and it can be computed very easily.

### 4.3 Remarks

1. Games of common agency (Bernheim and Whinston [4]) are of course a special case of the games we are considering, where $N = \{1\}$. In the case of an agent with two actions, the vector of weights $w^m(s)_{s \in S}$ of the principal $m$ is a scalar, and the condition (13) that they are balanced requires these scalars to the same. So a pure strategy equilibrium giving $\hat{s}$ as equilibrium outcome exists if and only if, for all $s$, $\sum_{m \in M}(G^m(\hat{s}) - G^m(s)) \geq 0$. An equilibrium in pure strategies with outcome $\hat{s}$ exists if and only if $\hat{s}$ is efficient.

2. The other extreme case is one principal and many agents, that is $M = \{1\}$. Balancedness means that

$$\sum_{s \in S} w^1(s)(G^1(\hat{s}) - G^1(s)) \geq 0.$$

   for any nonnegative vector $w(s)$. This is equivalent to $G^m(\hat{s}) - G^m(s) \geq 0$ for all $s$. Hence, again, an efficient equilibrium always exists, which is the result that Segal [17] obtains in absence of agent interdependences.

3. Given a deviation $\bar{s}$, a possible vector of balanced weights is, for every $m \in M$,

$$w^m(s) = \begin{cases} 1 & \text{if } s = \bar{s}; \\ 0 & \text{otherwise.} \end{cases} \qquad (23)$$

   This vector is balanced because each principal is asking exactly the same deviation from all agents. Then, we get that a game is balanced only if, for every $\bar{s} \in S$,

$$\sum_{m \in M} (G^m(\hat{s}) - G^m(\bar{s})) \geq 0$$

   that is, $\hat{s}$ is the efficient action. This is an indirect way of getting to Proposition 2. Of course, efficiency does not in general imply balancedness. That is because the weights in (23) assume that all principals have the same `best' deviation and that need not be true.

4. Let $C(s) \equiv \sharp\{s : s_n \neq \hat{s}_n\}$, and $NW$ denote the set of balanced weights, normalized by
$$\sum_{s \in S} w^m(s) C(s) = 1.$$
Since the inequality defining a balanced game is homogeneous, the condition
$$\min_{w \in NW} \sum_{m \in M, s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) \geq 0$$
is necessary and sufficient for existence of an equilibrium in pure strategies giving $\hat{s}$ as action profile outcome. Now let
$$C \equiv \{(w^m(s))_{m \in M, s \in S} : w^m(s) \geq 0 \text{ for every } m, s; \text{ and } \sum_{s \in S} w^m(s) C(s) = 1\}.$$
By proposition (3), $NW \subset C$, and therefore a sufficient condition for the existence of equilibria in pure strategies is
$$\min_{w \in C} \sum_{m \in M, s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) \geq 0$$
But the set $C$ has a product structure: a vector $w$ belongs to $C$ if and only if each $m$-th component satisfies a set of constraints that only depend on $w^m$. The minimization problem is equivalent to:
$$\sum_{m \in M} \min_{s \in S} \left( \frac{G^m(\hat{s}) - G^m(s)}{C(s)} \right) \geq 0, \tag{24}$$
so we may state:

**Proposition 4** An equilibrium in pure strategies giving $\hat{s}$ as equilibrium action profile exists if (24) holds.

5. Pérez-Castrillo [14] studies a game with multiple principals (firms) and multiple agents (workers) in which the profit of a firm depends in a general way on which workers it hires. Firms make salary offers to workers, who simply maximize their income. Thus, Pérez-Castrillo's game can be written as a GPTA in which, for agent $n$, the space of possible actions consists of choosing for which principal they will work for: $S_n = M \cup \emptyset$. Moreover, the payoff of a principal depends on whether an agent works for her but, if he does not work for her, it does not depend on which competitor the agent works for. In this game, Pérez-Castrillo shows the equivalence between (1) the set of subgame-perfect equilibria and (2) the set of stable solutions of a cooperative game in which coalitions of many agents and one principal can form.

   Unfortunately, without the restriction on $S_n$ imposed by Pérez-Castrillo, concept (2) is not well defined. The choice of each agent cannot be reduced to the choice of which principal he is associated with. If the reduction is not possible, then the value of a coalition is not given, but depends on what principals and agents outside the coalition will do. For instance, suppose there are three principals $(A; B; C)$ and two agents (1, 2) and that Agent 1 can work for Principal A or B while Agent 2 can work for Principal B or C. But — contrary to Pérez-Castrillo — suppose that Principal A cares whom Agent 2 works for. Then, it is impossible to give a value to coalition $\{A; 1\}$ unless it is known what the other principals and agents are doing. The same problem would apply to other coalitional concepts that one could think of using instead of (2).

6. Related to the previous point, one may wonder what the connection is between trasferable utility games (TUG) and games played through agents. We can show that every TUG can be put in correspondence with a (very simple) GPTA and that the core of the TUG is nonempty if and only if the corresponding GPTA has a pure-strategy equilibrium.

   We begin by recalling some basic notions. Let $N$ be a finite set of players, and $v : 2^N \to \mathbb{R}$ a value function. This function associates to each coalition of players the value (or utility) that such coalition can get. The core of the game $(N; v)$ is the set of allocations $x \in \mathbb{R}^N$ such that:

   (a) (Group rationality) $\sum_{n \in N} x^n = v(N)$;
   (b) (Coalition rationality) for all $I \subseteq N$, $\sum_{n \in I} x^n \geq v(I)$.

   The Shapley-Bondareva theorem states that the core of $(N; v)$ is non-empty if and only if for every set of non-negative weights $(\lambda(I))_{I \subseteq N}$ that are balanced, that is that satisfy:

   $$\text{for every } n \in N; \quad \sum_{I : n \in I} \lambda(I) = \lambda(N)$$

   the following inequality holds:

   $$\sum_{I \subset N} \lambda(I) v(I) \leq \lambda(N) v(N):$$

   We will call these weights Shapley-Bondareva weights.

   The following definition (cfr Pérez-Castrillo [14, Section 3]) will be used to provide a link between the core and the set of equilibria of a very special class of GPTA's:

   **Definition 5** The GPTA induced by the TUG $(N; v)$ is given by: $M = \{1; 2\}$, $N$, $S^n = M$ for every agent, and payoff

   $$G^m(s) = v(I_m(s))$$

   for every principal, where $I_m(s)$ is the set of agents "choosing" the principal $m$, namely:

   $$I_m(s) \equiv \{n : s^n = m\}:$$

We assume that $v$ is superadditive (that is, for any set $I \subseteq N$, and subsets $J \subseteq I$,

$$v(I) \geq v(J) + v(I \setminus J).$$

The next theorem provides a non-cooperative foundation for the core solution concept, based on GPTA's.

**Theorem 4** An allocation $(\hat{x}^n)_{n \in N}$ is a core allocation of the game $(N; v)$ if and only if there is an equilibrium in pure strategies of the induced GPTA, where the equilibrium transfers $(\hat{t}^j_n)_{j \in M; n \in N}$ satisfy:

$$\hat{x}^n = \sum_{m \in M} \hat{t}^m_n(\hat{s}^n).$$

The theorem is illustrated through a well-known TUG with an empty core: the majority game. There are $N = 2K + 1$ players, and the value is:

$$\begin{aligned} v(J) &= 0 \text{ if } \#J \leq K; \\ &= 1 \text{ if } \#J \geq K + 1. \end{aligned}$$

A possible interpretation is that players have to divide one dollar, and any coalition with the simple majority can vote to itself the dollar. The core of this game is empty. The induced GPTA is the voting game presented earlier in this section (in which $x = 1$ so that the two principals are identical). The interpretation is that there is an assembly that can assign a dollar to exactly one of two principals. From Theorem 2, we easily see that this game has no pure-strategy equilibrium.[11]

While Theorem 4 is proven by establishing a direct connection between the core and a noncooperative equilibrium of the induced GPTA, one could also prove it indirectly by linking the Bondareva-Shapley weights with the weights used in Theorem 2. The relation between the two sets of weights is:

$$\lambda(N) = \sum_{J \subseteq N} w^1(s_J) - w^2(s_\emptyset);$$

$$\lambda(J) = w^1(s_J) + w^2(s_{N \setminus J}).$$

where for all $J \subseteq N$, $s_J$ is the outcome when agents in $J$ choose $s_n = 1$ and agents in $N \setminus J$ choose $s_n = 2$. In the majority game above, if $N = \{1; 2; 3\}$, a collection of Bondareva-Shapley weights that shows that the core is empty is:

$$\lambda(12) = \lambda(13) = \lambda(23) = 1/2;\ \lambda(123) = 1;$$

with all the other $\lambda$'s equal to zero, and a set of corresponding $w$ is:

$$w^1(s_{12}) = w^1(s_{23}) = w^2(s_2) = 1/2; w^m(s_J) = 0 \text{ for all other } m; J.$$

[11] Any game played through agents induced is nongeneric also because the two principals are identical. Hence, the link between efficiency and pure strategies is broken. For instance, in the voting game in which $x = 1$ any outcome is efficient and hence mixed-strategy equilibria are efficient too.

## 5 The General Case

We now leave the simplified environment where agents have only two actions and are not directly affected by the action they choose. We thus revert to the general model introduced in Sections 2 and 3.

If agents have more than two actions, a pure-strategy equilibrium need not be efficient. This is already true if $N = \{1\}$ (common agency). Consider the example (see [4]) where $M = \{1; 2\}$, $N = \{1\}$, $\#S_1 = 4$, and $F_1(s_n) = 0$, with

$$G^1 = (8; 6; 0; 1); G^2 = (0; 6; 7; 1);$$

Here $t^1 = (7; 0; 0; 0)$, $t^2 = (0; 0; 7; 0)$, and $s = 1$ is a pure strategy equilibrium with an inefficient outcome. The main feature of this equilibrium is a failure of the two principals to coordinate on the efficient action. Principal 1 does not make an offer on action 2 because Principal 2 is not making an offer either, and viceversa. There exists another pure-strategy equilibrium in which $t^1 = (3; 1; 0; 0)$, $t^2 = (0; 2; 3; 0)$, and $s = 2$, which selects the efficient action and gives a higher payoff to both principals.

To overcome this multiplicity of equilibria, in common agency Bernheim and Whinston introduce the notion of truthful transfers. A transfer vector is truthful if, for all actions, it is equal to the principal's gross payoff minus a constant (save for the nonnegativity constraint on transfers). Formally,

**Definition 6** If $N = \{1\}$, a transfer vector $t^m$ is truthful relative to $\hat{s}$ if for every $s \in S$

$$t^m(s) = \max(0; t^m(\hat{s}) + G^m(s) - G^m(\hat{s}))$$

A pure strategy equilibrium giving $\hat{s}$ as equilibrium action is truthful if the strategy of every principal is truthful relative to $\hat{s}$.

In common agency, truthful equilibria play a fundamental role. They always exist, the equilibrium action outcome is efficient ([4, Theorem 2]) and they are coalition proof ([4, Theorem 3]). The intuition is that truthful transfers restrict offers on out-of-equilibrium actions not too be too low with respect to the principals' payoffs and therefore exhausts all gains from coalitional deviations.

But truthful equilibria are very hard to come by if there is more than one agent. For instance, in the prisoner's dilemma game a vector of truthful strategies should satisfy (disregarding the nonnegativity constraints) $a = y - x; b = y$ for the first principal, and $a = y; b = y - x$ for the second, and this is impossible. Intuitively, the requirement of being truthful imposes too many equations on the strategy.

However, one can relax Definition 6 from equality to inequality. A weaker condition is that for every $s \in S$, $t^m(s) \geq t^m(\hat{s}) + G^m(s) - G^m(\hat{s})$, or alternatively

$$G^m(s) - t^m(s) \geq G^m(\hat{s}) - t^m(\hat{s}):$$

This definition maintains the feature that offers on out-of-equilibrium actions cannot be too low and it can be extended to a GPTA with many agents:

**Definition 7** In a GPTA, $t^m$ is weakly truthful relative to $\hat{s}$ if

(WT) For every $m \in M$ and $s \in S$, $G^m(\hat{s}) - \sum_{n \in N} t_n^m(\hat{s}^n) \geq G^m(s) - \sum_{n \in N} t_n^m(s_n)$.

A weakly truthful equilibrium is a pure strategy equilibrium giving $\hat{s}$ as equilibrium outcome, and in which the strategy of every principal is weakly truthful relative to $\hat{s}$.

A consequence of this definition is that – like truthful equilibria of common agency games – weakly truthful equilibria of GPTA's are always efficient:

**Proposition 5** The outcome of a weakly truthful equilibrium is efficient.

The pair of action and transfer outcomes of a weakly truthful equilibrium has a simple characterization. The necessary and sufficient conditions for an action profile to be supported by a weakly truthful equilibrium are the same as those for an action profile to be supported by an equilibrium (Theorem 1), except that (IC) is substituted with the stronger requirement that transfers be weakly truthful:

**Proposition 6** A pair $(\hat{t}, \hat{s})$ of transfers and action profiles is the outcome of a weakly truthful equilibrium if and only if they satisfy (WT), (AM) and (CM).

To check that the definition of weakly truthful equilibrium is consistent with the analysis of the previous section, consider what happens to weakly truthful equilibria if each agent has only two actions and cares solely about monetary payoff. In this case, it is immediate to check that (AM) and (IC) imply (WT) and, by Proposition 6:

**Corollary 2** if each agent has only two actions and cares solely about monetary payoff, then all equilibria are weakly truthful.

Weak truthfulness eliminates inefficient equilibria that are due to coordination problems. If each agent has only two actions, coordination problems do not arise because each principal will contribute for either one action or the other. Hence, weak truthfulness has no bite.

We now move to the question of whether a weakly truthful equilibrium exists. As in the previous section, we pose this question with respect to a particular action profile, that is, we ask whether, given $\hat{s} \in S$, there exists a weakly truthful equilibrium that produces outcome $\hat{s}$. We need to redefine balancedness:

**Definition 8** In a GPTA, the vectors $w$ and $z$ with respective dimensions $MS$ and $H$ are said to be vectors of balanced weights if all their elements are nonnegative, and

$$\text{for every } m \in M; n \in N; a_n \in S_n \setminus \hat{s}_n; \quad \sum_{\{s: s_n = a_n\}} w^m(s) = z_n(a_n):$$

A GPTA is balanced if and only if for every pair of vectors of balanced weights $w$ and $z$ we have:

$$\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n \in N} \sum_{s_n \in S_n} z_n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) \geq 0$$

If agents have only two actions, there is only one possible deviation for each agent. With more than two actions, the condition that principals put the same sum of weight must hold for every agents and for every deviation that the agent has. Moreover, the definition of balancedness now includes weights on agents as well as principals. This is because, if we are considering a deviation from $\hat{s}$ to $s$, we have to take into account not only the benefit of principals but also that of agents. A game is balanced (with respect to a given action $\hat{s}$) if, for any vector of balanced weights, the sum of the direct change in payoffs for principals and agents of any possible deviation is negative. If agents do not care about actions and there are only two actions per agent, we recover the definition of balancedness used in the previous section.

The main result of this section is:

**Theorem 5** *A GPTA with agent preferences has a weakly truthful equilibrium with outcome $\hat{s}$ if and only if it is balanced with respect to $\hat{s}$.*

One question that is left open is whether there can be games that do not have a weakly truthful pure-strategy equilibrium but have a non-truthful pure-strategy equilibrium supporting the efficient outcome. The answer is positive, as illustrated by the following two-principal, two-agent, three-action-per-agent example:[12]

| | L | C | R |
|---|---|---|---|
| T | $3, 0$ | $0, x$ | $-10, 0$ |
| M | $0, x$ | $0, x$ | $-10, 0$ |
| B | $-10, 0$ | $-10, 0$ | $-10, 0$ |

By applying Theorem 5, we see that this game has no weakly truthful equilibrium. To see that balancedness is violated, set $w^1(MC) = w^2(ML) = w^2(TC) = 0.5$ and all the other weights equal zero. Indeed, this game is just the Opposite Interest Game with the addition of a line and a column that are extremely bad for principal 1.

However, this game has a non-truthful pure-strategy equilibrium with outcome $(T, L)$. Actually, there is a continuum of them. One is as follows. Principal 1 offers $t_1^1(T) = t_2^1(L) = 5$ and zero on all other actions. Principal 2 offers $t_1^1(B) = t_2^1(R) = 5$ and zero on all other actions. As usual, agents maximize revenues, and, in case of indifference, select $(T, L)$. While somewhat unconvincing, this situation is an equilibrium because on one side any attempt by Principal 1 to save money would induce a payoff of $-10$, and on the other side Principal 2 finds it to expensive to deviate on $M$ or $C$.

This example shows that, with more than two actions, weak truthfulness – and hence balancedness – is only a sufficient condition for the existence of an efficient equilibrium.

## 6 Convexity and Bork's Claim

Balancedness as defined in cooperative game theory has a useful connection to convexity. Scarf [16] shows that a market game where agents have convex preferences has a nonempty core. This is achieved by proving that convexity is a sufficient condition for balancedness. A result in the same spirit – albeit with a different proof – can be derived for our definition of balancedness:

[12] We are grateful to Bruno Jullien for suggesting this example.

**Theorem 6** Assume that: For each agent $n$; the action space $S_n$ is a convex set in $\Re^{k_n}$, where $k_n$ is some natural number; For each agent $n$, the payoff function $F_n(s_n)$ is bounded, continuous, and concave in $s_n$;For each principal $m$, the payoff function $G^m(s)$ is bounded, continuous, and concave in $s$; There exists $\hat{s} \in \arg\max \sum_{n \in N} F_n(s_n) + \sum_{m \in M} G^m(s)$. Then, there exists a weakly truthful equilibrium with outcome $\hat{s}$.

Theorem 5 can be used to evaluate the validity of Judge Bork's [7] claim that unregulated vertical contracting leads to productive efficiency. To do that, we introduce the Vertical Contracting Game. Let $M$ be a set of sellers (upstream firms) and $N$ a set of buyers (final consumers or downstream firms). Let $q^m \in Q^m = [0; \bar{q}^m]$ be the quantity produced by $m$. Let $q = (q^1; ::::; q^M)$. The cost of production for $m$ is $C^m(q)$, where $C^m$ is assumed to be continuous, convex and bounded. The cost of production of $m$ may depend on $q^{-m}$ because sellers compete for the same inputs.

Let $q_n = (q_n^1; ::::; q_n^M) \in Q^1 \times \cdots \times Q^M$ denote the vector of quantities that buyer $n$ buys. The benefit (or revenue) that $n$ derives from $q_n$ is $B_n(q_n)$, and is assumed to be continuous, concave, and bounded in $q_n$.

Each seller offers a menu of contracts to each buyer. Let $p_n^m(q_n)$ the price that $m$ asks from $n$ if $n$ chooses vector $q_n$. This allows for nonlinear pricing and for exclusivity clauses. For instance, seller $m$ can impose an exclusivity clause on competitor $\bar{m}$ by setting $p_n^m(q_n)$ at a prohibitively high level when both $q_n^m$ and $q_n^{\bar{m}}$ are strictly positive (note that $p_n^m$ need not be continuous). However, the buyer can always choose not to buy. Hence, we impose the restriction that $p_n^m(q_n) \leq 0$ whenever $q_n^m = 0$.

The Vertical Contracting Game is not a GPTA as defined here because payments go from agents to principals. However, with a simple "trick" we can fit it in our framework. Instead of prices, we redefine contracts in terms of discounts from a prohibitively high linear pricing schedule.

Define a GPTA in which Principal $m$'s gross payoff is $G^m(q) = k \sum_{n \in N} q_n^m - C^m(q)$ and agent $n$'s is $F_n(q_n) = B_n(q_n) - k \sum_{m \in M} q_n^m$, where $k$ is a very large positive number. The action space of agents is the same of the Supply Contract Problem. Principals instead can offer nonnegative transfers to agents of the usual form $t_n^m(q)$. This new game fits the conditions of Theorem 5. Thus we know that it has a weakly truthful equilibrium $(\hat{t}; \hat{q})$ with outcome $\hat{q} \in \arg\max_q \sum_{n \in N} B_n(q_n) - \sum_{m \in M} C^m(q)$.

To revert to the initial Vertical Contracting Game, we have to make sure that the nonnegativity on constraint is never binding, which is true if in equilibrium $\hat{t}_n^m(q) \geq 0$ whenever $q_n^m > 0$. If this is not the case, we increase $k$ until we find a $k$ for which the nonegativity constraint is never binding (such a $k$ exists if we impose the technical assumption that the $R$'s and the $C$'s are differentiable and that the total differential is everywhere lower than a given number $l$).

Then, we can go from $(\hat{t}; \hat{q})$ to an equilibrium $(\hat{p}; \hat{q})$ of the Vertical Contracting Game by defining $\hat{p}_n^m(q_n) = k \sum_{n \in N} q_n^m - \hat{t}_n^m(q)$ for all $m$, $n$, and $q_n$. The restriction that $p_n^m(q_n) \leq 0$ whenever $q_n^m = 0$ is automatically satisfied. We have thus shown:

**Proposition 7** The Vertical Contracting Game has an equilibrium that maximizes the joint surplus of sellers and buyers, that is, in which the outcome $\hat{q} \in \arg\max_q \sum_{n \in N} R_n(q_n) - \sum_{m \in M} C^m(q)$.

To interpret Proposition 7, we need to know who the buyers are. If they are final consumers, then Proposition 7 says that unrestricted vertical contracting will lead to an outcome that is optimal from a Utilitarian perspective. If buyers are downstream firms, it tells us that all the firms involved in the game – upstream and downstream – behave as if they were owned by the same person. We would need to know how the downstream firms relate to consumers in order to say whether this outcome is in any sense efficient. However, if the game between downstream firms and consumers satisfies, in turn, the assumptions of Proposition 7, then we know that the outcome is again optimal from a Utilitarian perspective.

A crucial assumption behind Proposition 7, and hence behind Bork's claim, is that there are no direct externalities among buyers. This is a reasonable restriction if buyers are final consumers (except in the case of network goods) but not if buyers are downstream firms. In that case, there will be no externalities only under the stringent condition that the downstream firms operate on different markets.

The other important assumption is, of course, that the cost functions of sellers are convex. Violations occur if some sellers have to sustain fixed costs in order to produce the first unit or, more generally, if they face economies of scale, as in the Opposite Interest Game example.

## 7 Mixed-Strategy Equilibria

We have seen several examples, by now, showing that existence of equilibria is not ensured with pure strategies only. We provide here a general existence result. To do this, we define the problem as a game with endogenous sharing rules. (see Simon and Zame, [18]). We recall that an $M$-players game with an endogenous sharing rule is a tuple $\Gamma = (T^1, \ldots, T^M; Q)$ consisting of a strategy space $T^m$ for each player, and a payoff correspondence $Q : T^1 \times \ldots T^M \to \mathbb{R}^M$. A sharing rule is a Borel measurable selection from the correspondence $Q$; i.e. is a Borel measurable function $q : T \to \mathbb{R}^M$ such that $q(t) \in Q(t)$ for each $t \in T$. A solution for $\Gamma$ is a sharing rule and a mixed strategy profile such that, given the sharing rule, each player's action is a best response to the mixed strategy of the other players.

Simon and Zame [18] provide a general existence result for a large class of games with endogenous sharing rules that satisfy the following conditions:

1. there is a dense subset $T^*$ of the product of the strategy spaces $T$, and a bounded continuous function $\phi : S^* \to \mathbb{R}^M$;

2. $C_\phi$ is the correspondence whose graph is the closure of the graph of $\phi$, and $Q(t)$ is the convex hull of $C_\phi(t)$ for each $t$.

The following existence theorem is an application of the main theorem of Simon and Zame [18]:

**Theorem 7** The transfers game has a solution in mixed strategies.

A GPTA is a two-stage game. Given the principals' transfers, the second stage is a finite game played among the agents. Let $S(t)$ denote the set of second stage equilibria given $t$. It is easy to see that $S$ is nonempty. Moreover, with passive beliefs, for almost all $t$, $S$ is continuous in $t$ (because each agent strictly prefers one action over the others). We can view the principal stage as a game with M players in which S(t) is given. This game satis¯es Simon and Zame's conditions for a game with endogenous sharing rules and therefore is guaranteed to have an equilibrium. The equilibrium will include a sharing rule that dictates what each agent should do in case she is indi®erenet between two actions.

To illustrate mixed-strategy equilibria, reconsider the Opposite Interest Game of Section 4. For that game, we have shown that there does not exist a pure-strategy equilibrium. Theorem 7 guarantees the existence of a mixed strategy equilibrium. It turns out that there exists a mixed-strategy equilibrium as follows: principal 1 makes a transfer

$$(t; 0; t; 0) \text{ according to the CDF } G(t) = \frac{x \text{ ¡ } \frac{3}{2}}{x \text{ ¡ } t}$$

with t 2 [0; 3=2]. The ¯rst principal makes with probability $\frac{1}{2}$ a transfer

$$(0; s; 0; 0) \text{ according to the CDF } F(s) = \frac{s}{3 \text{ ¡ } s}$$

and with probability $\frac{1}{2}$ a transfer

$$(0; 0; 0; s) \text{ according to the CDF } F(s) = \frac{s}{3 \text{ ¡ } s};$$

in both cases with s 2 [0; 3=2].

# 8 Sequential Versions

In the trasfer game that has been considered so far, all principals play at the same time and all agents play at the same time. One may suspect that it is this simultaneity that drives the ine±ciency results. We devote this section to sequential variations of the original game with the goal of showing { by means of examples { that our ine±ciency results are robust. We ¯rst look at the timing in which principals make their o®er sequentially (principal sequentiality) and then move to agents choosing their actions sequentially (agent sequentiality).

Of the three applications discussed in the introduction: lobbying has been modeled both in a simultaneous fashion (e.g. Dixit, Grossman, and Helpman [8]) and in a principal-sequential manner (Groseclose and Snyder [10]); vertical contracting been studied in the simultaneous framework (Bernheim and Whinston [5]) and in a principal-sequential framework (e.g. Aghion and Bolton [1]) with the idea that one seller is the incumbent and enjoys a ¯rst-mover advantage; two-sided matching has been studied in a simultaneous manner. The agent-sequential version could capture a lobbying process with separate powers or vertical contracting with buyers arriving sequentially (large companies running procurement auctions).

## 8.1 Principal Sequentiality

Agina, we make use of the Opposite Interest Game. Let us start with the case in which Principal 1 moves ¯rst, she makes o®ers to both agents. Then, Principal 2 observes the o®ers and makes her o®ers to the two agents, who, ¯nally, choose their actions. With this ordering, the e±cient outcome $(T; L)$ never arises in a subgame perfect equilibrium. This is immediate because, in order to guard from the threat of Principal 2 bribing one of the two agents, principal 2 must o®er at least $2x$, which is more than 3. If, on the other hand, Principal 1 moves second, then the e±cient outcome will arise for sure. This is because Principal 2 is willing to o®er at most $x$ and only to one of the two agents, and Principal 1 is willing to counter such an o®er.

Thus, one may conjecture that for every GPTA played in a principal-sequential fashion there exists an ordering of principals such that the game has an e±cient subgame-perfect equilibrium. However, this conjecture is incorrect, as the following example shows. Each agent has three actions and the payo® function is:

| | L | C | R |
|---|---|---|---|
| T | $3; 3$ | $0; 5$ | $5; 0$ |
| M | $0; 5$ | $0; 0$ | $0; 0$ |
| L | $5; 0$ | $0; 0$ | $0; 0$ |

For both possible ordering of principals, if the e±cient outcome $(T; L)$ were supported by a SPE, the ¯rst moving principal would have to pay at least $2+2$ to guard against deviations by the second mover, but this is clearly not optimal because she can always get at least zero.

## 8.2 Agent Sequentiality

In the agent-sequential version of the Opposite Interest Game, the two principals simultaneously choose their transfers to Agent 1, who then chooses his action. The principals observe Agent 1's choice and make o®ers to Agent 2, who in turn takes an action. Given the symmetry of the game, things would not change if Agent 2 were to go ¯rst.

The outcome of a subgame perfect equilibrium of the agent-sequential version of the Opposite Interest Game cannot be $(T; L)$. To see this, examine the two possible subgames of the second stage (for simplicity, assume $x = 2$). If Agent 1 has chosen $B$, then in the second stage both principals are indi®erent as to what Agent 2 chooses. If Agent 2 has chosen $T$, then the second stage is an auction in which principal 1 has valuation 3 and Principal 2 has valuation 2. Principal 1 wins and must pay at least 2. Thus, we can substitute the second stage payo®s in the ¯rst stage. If Agent 1 chooses $T$, the payo®s are $(0; 1)$. If he chooses $B$, they are $(2; 0)$. Thus, the ¯rst stage is an auction in which Principal 1 has valuation 1 and Principal 2 has valuation 2. Principal 2 must win, which shows that the outcome of a SPE cannot be $(T; L)$.

This example shows that the ine±ciency that is present in simultaneous GPTA's not only persists in the agent-sequential case but it can also become more severe. In the Opposite Ineterest Game, in the simultaneous case the outcome is $(T; L)$ with positive probability. In the sequential case, it is never $(T; L)$ and, thus, it is always ine±cient.

It is interesting to contrast these observations with Bergemann and VÄlimÄki [3]. They consider the dynamic version of common agency, in which a set of principals face the same agent repeatedly. At each time, principals make o®ers to the agent contingent on the action that the agent chooses at the present time. The agent's and principals' payo® functions depend on time, the current action chosen by the agent, and all the previous actions chosen by the agent. Thus, in making their o®ers, principals must take into account the future e®ect of the current action chosen by the agent. Similarly, in choosing the current action, the agent does not only take into account current contributions and current direct payo®, but he must also consider the future e®ect of the current action. Bergemann and VÄlimÄki prove that all Markov-perfect equilibria in truthful strategies of the dynamic common agency game are e±cient.

The di®erence between Bergemann and VÄlimÄki's dynamic common agency and an agent-sequential GPTA is that in their model there is a unique agent. Let us look at a modi¯cation of the example above that ¯ts Bergemann and VÄlimÄki's framework. Suppose that the payo® is as above. The game is played in an agent-sequential manner. However, let us now suppose that Agent 1 and Agent 2 are the same person, so that there is a unique agent who maximizes the undiscounted sum of the transfers received at time 1 and at time 2. Indeed, this modi¯ed version of the example above has an e±cient equilibrium. If the agent chooses T in time 1, then he gets a transfer of 2 at time 2. If the agent chooses B at time 1, then he gets a payo® from transfers of 0 at time 2. At time 1, Principal 2 must o®er at least 2 more than Principal 1 in order to induce the agent to choose B. There exists a subgame perfect equilibrium in which $t^1_1(T) = 0$, $t^2_1(B) = 2$, $t^1_2(L) = t^2_2(R) = 2$, and the agent chooses $(T; L)$.

## 9 Outcome-Contingent Contracts

Throughout the paper we have assumed that a transfer between Principal m and Agent n can be conditional on the action taken by the agent $s_n$. However, the payment could also depend on the whole outcome s rather than only on the component under the control og Agent n. In this section, we discuss this possibility and we show that this addition does not solve the problem of non-existence of e±cient equilibria.

A GPTA with outcome-contingent contracts is de¯ned as in Section 2 except that the transfer o®ered by Principal m to Agent n is now dependent on s rather than $s_n$ (we call it ¿$^m_n$(s)). In general, we may expect the set of equilibria to be modi¯ed because the conditions for agent maximization are changed. Now, the agent needs to know what other agents are doing before deciding what he should do. In this sense, there is a parallel between a GPTA with outcome-contingent contracts and a GPTA with externalities as studied in Section 11. We can expect that, also in the case with outcome-contingent contracts, agent beliefs will play a crucial role.

The ¯rst thing to note is that a characterization of pure-strategy equilibria on the line of Theorem 1 is not possible anymore. What cannot be written in a simple form are the (IC) conditions. With action-contingent contracts, principal m knows exactly how much money it takes to convince agent n to deviate from candidate equilibrium action $\hat{s}_n$ to alternative action $s_n$. With outcome-outcome contingent contracts, this depends on what agent n expects other agents to do, which in turn depends on what the other agents expect n to do.

However, in the spirit of the paper, we would like to know whether outcome-contingency is su±cient to restore e±ciency { that is, whether for any GPTA with outcome-contingent contracts there exists an equilibrium supporting an e±cient outcome. The answer is negative. There are still games in which all equilibria are ine±cient. To see this, we can utilize once more the Opposite-Interest Game:

**Proposition 8** The Opposite-Interest Game, played with outcome-contingent contracts, has no e±cient equilibrium

Nevertheless, there are games in which the possibility of making outcome-contingent transfers creates e±cient, if somewhat unplausible, equilibria that do not exist if transfers are action-contingent. The point is illustrated by a variation of the Opposite Interest Game:

| | L | R |
|---|---|---|
| T | 3; 0 | 0; 2 |
| B | 0; 2 | 0; ¡ 100 |

in which the only di®erence is that Principal 2 receives a very negative payo® if both agents deviate from $(T; L)$. The action-contingent version of this game does not have a pure-strategy equilibrium. This can be checked through the balancedness condition, using the weights $w^1(BR) = w^2(BL) = w^2(TR) = 0{:}5$. Instead, the outcome-contingent version does have a pure-strategy equilibrium with outcome $(T; L)$. Principal 1 o®ers $¿^1_1(BR) = ¿^1_2(BR) = 10$ and zero for all other outcomes. Principal 2 o®ers zero on all outcomes. Agents 1 and 2 face a coordination problem. We assume they coordinate on $(TL)$ if and only if Principal 2 o®ers zero for all outcomes. Otherwise they coordinate on $(BR)$. Of course, if Principal 2 were to o®er more than 10 for either $(BL)$ or $(TR)$, she could get either of those outcomes, but such a deviation is not in her interest. Any lower deviation would be detrimental because it would induce the agent to coordinate on the very negative outcome $(BR)$.

## 10 Conclusions

The main lesson of this analysis is that in games played by agents there may exist a type of strategic ine±ciency that is absent when either there is only one agent (Bernheim and Whinston [4]) or there is only one principal (Segal [17]). Balancedness is a necessary and su±cient condition for e±ciency, and convexity is a su±cient condition. However, there are relevant economic situations in which the strategic ine±ciency is present and robust to modi¯cations of the game.

We have already discussed some possible extensions, that need more work: sequential timing, outcome-contingent contracts, and direct preferences. There is another topic that could prove interesting: the connection between GPTA's and auction theory. A GPTA has an alternative interpretation as a round of multiple auctions. Each agent is an auctioneer selling o® an action (which can be seen as the allocation of one or more objects { see Bernheim and Whinston [4]). Each principal is a bidder with preferences over the possible allocations.

Multiple auctions with a common set of bidders are of great practical importance. Many countries are presently engaged in the sale of UMTS licenses (third generation mobile phones). The national governments are the auctioneers and they act independently of each other. Large communication companies are the bidders and most of them are present in more than one country. Thus, the players and the preferences can be described as in a GPTA. However, what is di®erent is that in a GPTA the sale mechanism is a ¯rst-price sealed bid, while countries typically use more complex auction formats (another di®erence lies in the agent-sequential timing). The question that may be asked is under what conditions autioneers lose money and e±ciency by not coordinating with each other.

# 11 Appendix A: Agents with Direct Externalities

We now remove the restrictions that each agent has no direct interest in what the other agents are doing. Those instead of using $F_n(s_n)$, we write the agent's utility as $F_n(s)$.

As we know from Segal [17], we have no reason to expect that an efficient outcome will arise. The goal of this section will be to show that our machinery based on balancedness still applies to this case, although in general it will only tell us whether a pure-strategy equilibrium exists.

Agents without direct externalities do not face coordination problems, in the sense that the best response set of one agent does not depend on the action taken by other agents. This is not true anymore and the set of equilibria becomes much more complex. To overcome this problem, the literature with one principal and many agents has followed two main avenues, which are discussed in Segal [17]. The first is to assume that transfers are publicly observable and, in case a coordination problem arises, agents choose the outcome preferred by the principal. Clearly, this modeling strategy cannot be used when there are multiple agents. The second avenue – which is taken here – is to look at secret transfers and passive beliefs.

Transfers are secret. An agent observes the transfers offered to him but not the transfers offered to the other agents. Agent n thus observes $t^n \equiv \{t_n^m(s_n)\}_{(m \in M; s_n \in S_n)}$ and forms beliefs $\tau(t^n)$ about the transfers made to the other agents. We simplify the problem by restricting beliefs to be passive:[13]

**Definition 9** Agents hold passive beliefs if $\tau(t^n)$ is independent of $t^n$. A passive belief equilibrium is a pure-strategy perfect Bayesian equilibrium of the GPTA in which agents hold passive beliefs.

If agent n holds passive beliefs and he observes a deviation from one of the principals, he assumes that the principal has deviated only with him but is still offering the same transfers to the other agents. A passive belief equilibrium is then simply a pair $(\hat{t}; \hat{s})$.

Both the assumptions of private offers and passive beliefs can be seen as arbitrary. Other assumptions are possible, and maybe more appropriate in certain circumstances. We do not aspire to provide a comprehensive treatment of multi-principal multi-agent games with externalities among agents. Our goal in this section is simply to show that the logic of the results we have obtained in the previous sections can be fruitfully extended to the case in which agents have direct preferences.

Let a weakly truthful passive belief equilibrium be a passive belief equilibrium in which (WT) is satisfied (as (WT) does not depend on the agents' preferences, Definition 6 applies this section as well). Let $(s_n; \hat{s}_{-n}) \equiv (\hat{s}_1; \ldots; s_n; \ldots; \hat{s}_N)$ be the outcome when all agents play $\hat{s}$ but agent n deviates to $s_n$. We can now show that the equilibrium characterization given in the previous sections is still valid, albeit with small modifications:

**Theorem 8** A pair $(\hat{t}; \hat{s})$ of transfers and action profiles is the outcome of a weakly truthful passive belief equilibrium if and only if the following conditions are satisfied:

(AM') for every $n \in N$, $s_n \in S^N$,

$$\sum_{m \in M} \hat{t}_n^m(\hat{s}_n) + F_n(\hat{s}) \geq \sum_{m \in M} \hat{t}_n^m(s_n) + F_n(s_n; \hat{s}_{-n});$$

[13] See McAfee and Schwartz [13] for a discussion of secret transfers and further references.

(IC') for every $m \in M$, $s \in S$,

$$G^m(\hat{s}) + \sum_{n\in N}\sum_{j\neq m}\hat{t}^j_n(\hat{s}_n) + \sum_{n\in N} F_n(\hat{s}) \geq G^m(s) + \sum_{n\in N}\sum_{j\neq m}\hat{t}^j_n(\hat{s}_n) + \sum_{n\in N} F_n(s_n, \hat{s}_{-n});$$

(CM') for every $m \in M$, $n \in N$,

$$\sum_{j\in M}\hat{t}^j_n(\hat{s}^n) + F_n(\hat{s}) = \max_{a\in S_n}\Big[\sum_{j\in M}\hat{t}^j_n(a) + F_n(a, \hat{s}_{-n})\Big].$$

Proof: The proof is exactly the same as the proof of Theorem 1, except that $T^m_n(s_n) \equiv \sum_{j\neq m}\hat{t}^j_n(s_n) + F_n(s_n)$ is replaced with $T^m_n(s_n) \equiv \sum_{j\neq m}\hat{t}^j_n(s_n) + F_n(s_n, \hat{s}_{-n})$. ∎

As a limit case, consider what happens to the conditions of Theorem 8 when there are no principals ($M = \emptyset$). The conditions (AM'), (IC'), and (CM') reduce to: for every $n \in N$ and $s \in S$, $F_n(\hat{s}) \geq F_n(s_n, \hat{s}_{-n})$. This is the necessary and sufficient condition for $\hat{s}$ to be the outcome of a Nash-equilibrium in the game played among agents.[14]

As in the previous sections, we study conditions for existence of pure-strategy equilibria – with the caveat that the link between pure strategies and efficiency is broken. First, we need to redefine balancedness. The definition of balanced weights is like in Definition 8, while the definition of balanced game is only slightly modified:

**Definition 10** A GPTA with externalities is balanced if and only if for every pair of vectors of balanced weights $w$ and $z$ we have:

$$\sum_{m\in M}\sum_{s\in S} w^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n\in N}\sum_{s_n\in S_n} z_n(s_n)(F_n(\hat{s}) - F_n(s_n, \hat{s}_{-n})) \geq 0$$

The main result of this section is:

**Theorem 9** A GPTA with agent preferences has a weakly truthful passive belief equilibrium if and only if it is balanced.

Proof: The proof is identical to the proof of Theorem 5 except that $F_n(s_n, \hat{s}_{-n})$ substitutes $F_n(s_n)$. ∎

When agents have no direct preferences, it is not true anymore that balancedness implies efficiency. There can exist a weakly truthful equilibrium which is not efficient. This, in turn, implies that weakly truthful equilibria are not generically unique anymore. The following example illustrates both these facts:

[14] It is also easy to see that, if we take a transfer game and let all the G's tend to zero uniformly (while keeping the F's constant), we obtain that the limit of (AM'), (IC'), and (CM') is the Nash condition. This suggests a degree of continuity between the concept of weakly truthful passive belief equilibrium and the concept of Nash equilibrium. This property need not hold if we assume more sophisticated beliefs which may introduce some implicit ability to coordinate.

**Example** Consider a game with two agents, two principals, and two actions per agent. Each agent can say "red" or "blue". The agents' preferences are: $(F_1(r;r) = 1; F_2(r;r) = 1)$; $(F_1(r;b) = 0; F_2(r;b) = 0)$; $(F_1(b;r) = 0; F_2(b;r) = 0)$; $(F_1(b;b) = 2; F_2(b;b) = 2)$. The principals' preferences are: $(G^1(r;r) = x; G^2(r;r) = 0)$; $(G^1(r;b) = 0; G^2(r;b) = 0)$; $(G^1(b;r) = 0; G^2(b;r) = 0)$; $(G^1(b;b) = 0; G^2(b;b) = x)$, where $x \geq 0$. Hence, the efficient outcome is $(r;r)$. It is easy to see that, if $x \leq 2$, both $(b;b)$ and $(r;r)$ satisfy balancedness and, hence, there are two weakly truthful equilibria outcomes. Agents are playing a pure coordination game with two equilibria: one efficient and one inefficient. When $x > 2$, the equilibrium with $(b;b)$ disappears. Only if principals have enough interest in the game, the inefficient equilibrium disappears.

However, there exists a simple (but strong) sufficient condition to restore efficiency and uniqueness:

**Corollary 3** If there exists an efficient action $s^*$ such that, for all $n \in N$ and all $s \in S$, $F_n(s_n^*; s_{-n})$ does not depend on $s_{-n}$, then all weakly truthful passive belief equilibria are efficient.

Proof: Let $\hat{s}$ be the outcome of a weakly truthful passive belief equilibrium. Consider a vector of balanced weights in which $w^m(s) = 1$ if $s = s^*$ and $z_n(s_n) = 1$ if $s_n = s_n^*$, with all the other weights equal to zero. For this set of weights, balancedness requires

$$\sum_{m \in M} \sum_{s \in S} (G^m(\hat{s}) - G^m(s^*)) + \sum_{n \in N} (F_n(\hat{s}) - F_n(s_n^*; s_{-n})) \geq 0$$

which, because of the assumption of the corollary, rewrites as

$$\sum_{m \in M} \sum_{s \in S} (G^m(\hat{s}) - G^m(s^*)) + \sum_{n \in N} (F_n(\hat{s}) - F_n(s^*)) \geq 0$$

This is true only if $\hat{s}$ is efficient. ■

The last corollary is the generalization of Segal [17, Proposition 3] to a multiple-principal environment. If there is only one principal, it is immediate to see that all equilibria are weakly truthful (or do not differ from a weakly truthful equilibrium in a payoff-relevant way). Then, Corollary 3 reduces to: If there exist an efficient action $s^*$ such that, for all $n \in N$ and all $s \in S$, $F_n(s_n^*; s_{-n})$ does not depend on $s_{-n}$, then all equilibria are efficient.

## 12 Appendix B: Principal-Sequential Version

Action sets of the agents and of the principals are defined as in the common agency game we have considered so far. But in the sequential common agency game principals move sequentially.

The $M$-th principal moves first and principal 1 moves last. Each principal announces a vector $t^m \in R^S$ of transfers to each agent; this announcement is commonly observed. Then the next principal does the same. Finally, agents move simultaneously choosing the action.

The equilibrium set can be characterized using the basic idea of principal agent problems. For every $m \in \{1,\dots,M\}$ and any vector of transfers $(t^k,\dots,t^M)$, we denote the subgame beginning after that vector of transfers has been announced by $\Gamma(t^k,\dots,t^M)$. For any such vector of transfers, and for every subgame-perfect equilibrium (SPE) of the game induced by this vector there is a set of actions chosen as equilibrium outcome of that subgame. We denote the set of all such actions as $\eta(t^k,\dots,t^M)$.

Take the principal $m$ who is moving in the subgame $\Gamma(t^{m+1},\dots,t^M)$. We may think that in solving the backwards induction problem this principal is choosing his transfer $t^m$ and the action profile of the agents, provided the choice of this pairs satisfies the incentive constraint that the chosen action is an equilibrium in the subgame beginning at $(t^m,\dots,t^M)$.

So $m$ is solving the problem:

$$\max_{s\in S}\Big(G^m(s)-\sum_{n\in N} t_n^m(s_n)\Big) \tag{25}$$

subject to the constraint that :

$$s \in \eta(t^m,\dots,t^M).$$

To study the equilibria of the game, consider first the principal who is moving last (that is, the principal 1). He is taking as given the vector $(t^2,\dots,t^M)$ of transfers of the previous principals, and is solving:

$$\max_{s_1\in S, t^1\in R^C}\Big(G^1(s_1)-\sum_{n\in N} t_n^1(s_n^1)\Big) \tag{26}$$

subject to:

$$s_1 \in \eta(t^1,\dots,t^M). \tag{27}$$

If we denote by $C(t^2,\dots,t^M;s_1)$ the minimum cost for principal 1 to implement $s_1$, that is the value of the problem:

$$\min_{t^1\in R^C}\sum_{n\in N} t_n^1(s_n^1);\ \text{subject to } s_1 \in \eta(t^1,\dots,t^M); \tag{28}$$

the problem of principal 1 is equivalent to

$$\max_{s_1} G^1(s_1) - C(t^2,\dots,t^M;s_1). \tag{29}$$

But $s_1 \in \eta(t^1,\dots,t^M)$ if and only if $s_n^1$ maximizes the payoff to agent $n$ for every $n$, that is if and only if:

$$F_n(s_n^1)+\sum_{j=1}^{M} t_n^j(s_n^1) \geq F_n(s_n')+\sum_{j=1}^{M} t_n^j(s_n') \tag{30}$$

for every $s_n' \in S_n$ and every $n$. Hence it is easily seen that:

$$C(t^2,\dots,t^M;s_1)=\sum_n\left[\max_{s_n'}\left(F_n(s_n')+\sum_{j=2}^{M} t_n^j(s_n')\right)-\left(F_n(s_n^1)+\sum_{j=2}^{M} t_n^j(s_n^1)\right)\right]. \tag{31}$$

The term $\sum_n \max_{s'_n}\left(F_n(s'_n) + \sum_{j=2}^M t^j_n(s'_n)\right)$ is a constant in $s^1$, and therefore the set of actions chosen at a SPE by the principal 1 in the subgame $\Gamma(t^2,\ldots,t^M)$ is the set of solutions of

$$\max_{s^1}(G^1(s^1) + \sum_{n\in N}(F_n(s^1_n) + \sum_{j=2}^{M} t^j_n(s^1_n))). \qquad (32)$$

In the case of the sequential game with a single agent, the reasoning above extends to all the principals. In fact one can prove:

**Proposition 9** For any $m$ and any $(t^{m+1},\ldots,t^M)$, the action $s_m$ is a solution of the problem:

$$\max_{s^m}(\sum_{j=0}^{m} G^j(s_m) + \sum_{j=m+1}^{M} t^j(s_k)) \qquad (33)$$

if and only if $s^m \in \eta(t^{m+1},\ldots,t^M)$ for a SPE of the game $\Gamma(t^{m+1},\ldots,t^M)$.

Here $t^j$ denotes the vector of transfers (one transfer for each action) to the single agent. A corollary of this proposition is obtained considering the case $m = M$. In this case the proposition implies that all the SPE equilibrium outcomes of the sequential game with one agent are efficient:

**Theorem 10** If there is only one agent, in any SPE the agent chooses the efficient action.

Details of the proof are given in Prat and Rustichini [15], who consider the single agent game extensively. Proposition 9 – and hence Theorem 10 –does not generalize to the case of many agents. To see why, consider the problem of Principal 2, the second to last to move. For a given vector $(t^3,\ldots,t^M)$ the problem of minimum cost to implement an action profile $\hat s$ is $\min_{t^2}\sum_{n\in N} t^2_n(\hat s_n)$ subject to the constraint that $\hat s$ solves (32), that is subject to:

$$G^1(\hat s) + \sum_{n\in N}(F_n(\hat s_n) + \sum_{j=2}^{M} t^j_n(\hat s_n)) \geq G^1(s') + \sum_{n\in N}(F_n(s^1_n) + \sum_{j=2}^{M} t^j_n(s'_n)) \qquad (34)$$

for every $s'$. This is a form different from the one in (30) for the last principal. So the minimum cost has a form different from (31). The formula for the minimum cost in a special case is given below.

To analyze the above problem, let:

$$F_n(s) \equiv G^1(s) + \sum_{n\in N}(F_n(s^1_n) + \sum_{j=3}^{M} t^j_n(s_n))$$

For the given vector $\hat s$ we denote, for each $s \in S$, $D(s) \equiv \{n \in N : s_n \neq \hat s_n\}$, and for any matrix $(F_n(s))_{s\in S}$

$$F_I \equiv \max_{\{s: D(s)\subseteq I\}} F_n(s).$$

**Lemma 1** The value of the cost minimization problem:

$$\min_{t\in R^C} \sum_{n\in N} t^n(\hat{s}_n) \tag{35}$$

subject to:

$$F(\hat{s}) + \sum_{n\in N} t^n(\hat{s}_n) \geq F_n(s) + \sum_{n\in N} t^n(s_n) \text{ for all } s \in S$$

is the same as the value of the problem

$$\min_{x\in R^N} x_N \tag{36}$$

subject to:

$$x_I \geq F_I - F(\hat{s}) \text{ for all } I \subseteq N; \tag{37}$$

where $x_I \equiv \sum_{n\in I} x_n$.

Note that the problem in (36) defines the least core of the game where the value of the coalition $I$ is $F_I - F(\hat{s})$.

**Proof.** Note first that if $\hat{t}$ is a solution of the problem (35), then so is the vector defined by

$$\begin{aligned} t^n(\hat{s}_n) &= \hat{t}(\hat{s}_n); \\ &= 0 \text{ if } s_n \neq \hat{t}_n. \end{aligned} \tag{38}$$

So we assume without loss of generality that the solution $\hat{t}$ of the problem (35) satisfies the condition (38). We call now $x_n$ the non-zero coordinate of the vector $\hat{t}$, that is $\hat{t}(\hat{s}_n)$. The problem (35) is therefore equivalent to

$$\min_x \sum_{n\in N} x_n;$$

subject to

$$F(\hat{s}) + \sum_{n\in N} x_n \geq F_n(s) + \sum_{\{n:s_n=\hat{s}_n\}} x_n; \text{ for all } s \in S. \tag{39}$$

Now observe that

$$\begin{aligned} \max_s (F_n(s) + \sum_{\{n:s_n=\hat{s}_n\}} x_n) &= \max_{\{I\subseteq N\}} (\max_{\{s:D(s)\subseteq I\}} F_n(s) + x_I) \\ &= \max_{\{I\subseteq N\}} (F_I + x_I); \end{aligned} \tag{40}$$

since $x$ is non-negative. So (39) is equivalent to

$$x^N + F(\hat{s}) \geq F_I + x_I \text{ for all } I \subseteq N$$

hence our claim. ■

In the case of the game with two principals from the previous discussion we have:

**Proposition 10** In the game with two principals and two agents, the action profile $\hat{s}$ is an equilibrium outcome if and only if it is the solution of

$$\max_{s^2} G^2(s^2) - \sum_n t_n^2(s_n^2) \tag{41}$$

subject to:

$$\sum_n t_n^2(s_n^2) \geq (G^1(s') + \sum_n F_n(s'_n)) - (G^1(s^2) + \sum_n F_n(s_n^2)) + \sum_n t_n^2(s'_n) \tag{42}$$

for every $s'$.

Consider now the problem of lemma (1) in the case of two agents. It is easy to see that (writing $a_I \equiv F_I - F(\hat{s})$) the problem

$$\min x^{12} \text{ sub. to } x^1 \geq a_1; x^2 \geq a_2; x^{12} \geq a_{12};$$

has value:

$$\max\{a_1 + a_2; a_{12}\}:$$

This gives an explicit solution to the cost minimzation problem of principal 2. Let for any matrix payoff $G^i$

$$RG(s_1; s_2) \equiv \max_{\{s' \in S^2\}} G^i(s_1; s'); \; CG^i(s_1; s_2) \equiv \max_{\{s' \in S^1\}} G^i(s_1; s_2);$$

and

$$MG^i(s_1; s_2) \equiv \max_{\{s' \in S^1; s'' \in S^2\}} G^i(s_1; s_2);$$

the matrices obtained taking the maximum along rows and columns and the overall maximum, respectively, and the matrix

$$B^i(s) \equiv \max\{MG^i(s); RG^i(s) + CG^i(s) - G^i(s):\}$$

The transfer of minimum cost for principal 2 among those that make principal 1 choose the action profile s is easily found to be, from lemma (1),

$$\max\{MG^1(s) - G^1(s); RG^1(s) + CG^1(s) - 2G^1(s):\}$$

An easy computation now shows that:

**Corollary 4** The action profile $\hat{s}$ is an equilibrium outcome of the sequential game if and only if it solves:

$$\max_s \left( \sum_n F_n(s_n) + G^1(s) + G^2(s) - B^1(s) \right) \tag{43}$$

In particular a sufficient condition for the equilibrium outcome to be efficient is that the strategic bias matrix $B^1(s)$ is constant.

**Example** If the payoff matrices are:

$$\begin{array}{ccc} & L & R \\ T & 3;0 & 0;2 \\ B & 0;2 & 0;2 \end{array} \tag{44}$$

then the matrix $B^1(s)$ for $G^1$ is

$$\begin{array}{ccc} & L & R \\ T & 3 & 3 \\ B & 3 & 3 \end{array}$$

so the equilibrium is efficient if principal 1 is the last to move. The equilibrium is inefficient if principal 2 is the last to move; in this case the $B^2$ matrix is

$$\begin{array}{ccc} & L & R \\ T & 4 & 2 \\ B & 2 & 2 \end{array}$$

The reason for the inefficiency is clear: when the principal 1 moves first, he will only make transfers, if any, on the first action of both agents. But the sum of these transfers can at most be 3, at equilibrium, since this principal can always insure a non-negative payoff. In particular, at least one of the two transfers must be less than 2. But then the principal 2 can get a gross payoff of 2, and a positive net payoff, rather than zero by beating such transfer.Core existence

The voting example provides a good motivation for a comparative discussion of the issue of existence of pure strategy equilibria in our games and the non-emptiness of the core.

We begin by recalling some basic notion. Let $N$ be a finite set of players, and $v : 2^N \to R$ a value function. This function associates to each coalition of players the value (or utility) that such coalition can get. The core is defined as follows.

# 13 Appendix C: Proofs

**Proof of Theorem 1:** Condition (AM) is clearly necessary and sufficient for the action $\hat{s}^n$ to be a best response of the agent $n$ to the transfers of the principals.

To prove the statement for the two remaining conditions, we characterize the best response of a principal $m$ to the given choice $(\hat{t}^j)_{j \neq m}$ of transfers of the other principals. To lighten the notation, we write:

$$T_n^m(s_n) \equiv F_n(s_n) + \sum_{j \neq m} \hat{t}_n^j(s_n):$$

Principal $m$ can induce from the agents the choice of any vector of actions $s \in S$ provided he promises a transfer greater than

$$(\max_{a \in S_n} T_n^m(a)) - T_n^m(s_n):$$

to agent n for the action $s_n$. Principal m will not choose $\hat{s}$ (and $(\hat{s}; \hat{t})$ is not an equilibrium) unless $\hat{s}$ solves

$$\max_{s \in S} G^m(s) - \sum_{n \in N} [(\max_{a \in S_n} T_n^m(a)) - T_n^m(s_n)]. \tag{45}$$

But $\sum_{n \in N} \max_{a \in S_n} T_n^m(a)$ is a constant independent of s, so $\hat{s}$ solves the problem (45) if and only if it satisfies (IC).

Finally we consider the condition (CM). The following lemma is very simple, but we state it for convenience. The proof is elementary.

Proof: For every action profile $\bar{s}$ and every vector $T^m$, the cost minimization problem in $t^m = (t_n^m(s_n))_{n \in N; s \in S} \geq 0$

$$\min_{t^m} \sum_{n \in N} t_n^m(\bar{s}_n) \text{ subject to } t_n^m(\bar{s}_n) + T_n^m(\bar{s}_n) \geq t_n^m(s_n) + T_n^m(s_n); \text{ for every } n \in N; s \in S \tag{46}$$

has value $c(\bar{s}; T^m)$ equal to $\sum_{n \in N} [(\max_{a \in S_n} T_n^m(a)) - T_n^m(\bar{s}_n)]$, and solution any $t_n^m(s_n) \geq 0$ such that:

$$t_n^m(s_n) = (\max_{a \in S_n} T_n^m(a)) - T_n^m(\bar{s}_n);$$

$$t_n^m(s_n) \leq (\max_{a \in S_n} T_n^m(a)) - T_n^m(s_n); \text{ for every } s_n.$$

■

If we apply the lemma choosing as $\bar{s}$ the candidate equilibrium action profile $\hat{s}$, we get that $t^m$ is a solution of the problem of minimum cost to implement $\hat{s}$ if and only if:

$$\hat{t}_n^m(\hat{s}_n) = \max_{a \in S_n} T_n^m(a) - T_n^m(\hat{s}_n) \tag{47}$$

and

$$\hat{t}_n^m(s_n) \leq \max_{a \in S_n} T_n^m(a) - T_n^m(s_n) \tag{48}$$

for every s. But (47), (48), and (AM) are equivalent to (47) and (AM). Since (47) is (CM), we have concluded our proof.

**Proof of Theorem 4** The equilibrium transfers are:

$$\text{for every } m; n: \ \hat{t}_n^m(m) = \hat{x}^n; \hat{t}_n^m(j) = 0 \text{ for } j \neq m;$$

and $\hat{s} = (1; \ldots; 1)$ (that is, all the agents choose the principal 1.) The condition

$$\sum_{n \in N} \hat{t}_n^m(m) = v(N)$$

must be satisfied (that is, principals are getting zero profit) because of the Bertrand nature of the competion among principals. And the condition

$$\text{for every } I \subseteq N; \sum_{n \in I} \hat{t}_n^m(m) \geq v(I)$$

must be satisfied, or the opposing principal might "buy" the coalition I by promising a larger total transfer to the agents in the coalition, and still make a positive profit.

**Proof of Corollary 1** Suppose for some n $\sum_{m\in M} t_n^m(\hat{s}_n) > 0$. Let m be one of the principals offering a strictly positive contribution on $\hat{s}_n$. If it were the case that $F_n(\hat{s}_n) + \sum_{m\in M} \hat{t}_n^m(\hat{s}_n) > F_n(\tilde{s}_n) + \sum_{m\in M} t_n^m(\tilde{s}_n)$, then m could save money by reducing $t_n^m(\hat{s}_n)$. The case $F_n(\hat{s}_n) + \sum_{m\in M} \hat{t}_n^m(\hat{s}_n) < F_n(\tilde{s}_n) + \sum_{m\in M} t_n^m(\tilde{s}_n)$ is pervented by (AM).

**Proof of Proposition 2** If we add (IC) over $m \in M$ we get that for every s:

$$\sum_{m\in M} G^m(\hat{s}) + (M-1)\sum_{n\in N}\sum_{m\in M} \hat{t}_n^j(\hat{s}_n) \geq \sum_{m\in M} G^m(s) + (M-1)\sum_{n\in N}\sum_{m\in M} \hat{t}_n^j(\hat{s}_n);$$

which, by (AM), implies $\sum_{m\in M} G^m(\hat{s}) \geq \sum_{m\in M} G^m(s)$.

**Proof of Proposition 5** Sum the inequalities (WT) over m. Sum the inequalities AM in Theorem 1 over n. Add the two resulting inequalities. The result is the inequality in (1), which defines efficiency.

**Proof of Proposition 6** Sum (AM) over n. To the resulting inequality, add (WT). The result is (IC). Hence, (WT) and (AM) imply (IC) and sufficiency is proven. Necessity is obvious because (AM) and (CM) are necessary by Theorem 1 and (WT) is necessary by the definition of weakly truthful equilibrium.

**Proof of Theorem 5** By Theorem 1, we can focus on (WT), (AM), and (CM), which is a system of inequalities and equalities. However, we can further simplify the problem by showing that there exists a solution to (WT), (AM), and (CM) if and only if there exists a solution to another system, which contains only inequalities:

**Proposition 11** There exists a weakly truthful equilibrium with outcome $\hat{s}$ if and only if there exists $d \in R^{MH}$ that satisfies:

(WTd) For all $s \in S$ and all $m \in M$, $\sum_{n: s_n \neq \hat{s}_n} d_n^m(s_n) \geq G^m(s) - G^m(\hat{s})$;

(AMd) For all $n \in N$ and all $s_n \in S_n$, $\sum_{m\in M} d_n^m(s_n) \leq F_n(s_n) - F_n(\hat{s}_n)$:

Proof: Let $d_n^m(s_n) \equiv t_n^m(s_n) - t_n^m(\hat{s}_n)$. Then, (WTd) is (WT) and (AMd) is (AM). Hence, the "only if" part is immediate.

To prove sufficiency, suppose that a matrix d has been found that satisfies (WTd) and (AMd). Clearly, there exists a nonnegative matrix $\hat{t}$ that satisfies (WT) and (AM). Starting from $\hat{t}$, we now construct a nonnegative matrix $t$ that satisfies (AM), (WT), and (CM).

In what follows, we hold n fixed: the procedure applies to any n. Start with $\hat{t}$ and define $t$ as follows. For m = 1 to M, let

$$b_n^m = \min\left\{\hat{t}_n^m(\hat{s}_n); F_n(\hat{s}_n) + \sum_{j=1}^{m-1} t_n^j(\hat{s}_n) + \sum_{j=m}^{M} \hat{t}_n^j(\hat{s}_n) - \max_{s_n \neq \hat{s}_n}\left(F_n(s_n) + \sum_{j=1}^{m-1} t_n^j(s_n) + \sum_{j=m+1}^{M} \hat{t}_n^j(s_n)\right)\right\};$$

$$t_n^m(s_n) = \max\{0; \hat{t}_n^m(s_n) - b_n^m\} \qquad \forall s_n \in S_n:$$

This definition implies

$$t_n^m(\hat{s}_n) = \hat{t}_n^m(\hat{s}_n) - b_n^m \tag{49}$$

We check that $\bar{t}$ satisfies (AM), (WT), and (CM).

(AM) is shown by induction. For all m, if

$$F_n(\hat{s}_n) + \sum_{j=1}^{m-1} t_n^j(\hat{s}_n) + \sum_{j=m}^{N} \hat{t}_n^j(\hat{s}_n) \geq F_n(s_n) + \sum_{j=1}^{m-1} t_n^j(s_n) + \sum_{j=m}^{N} \hat{t}_n^j(s_n) \tag{50}$$

then

$$F_n(\hat{s}_n) + \sum_{j=1}^{m} t_n^j(\hat{s}_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(\hat{s}_n) \geq F_n(s_n) + \sum_{j=1}^{m} t_n^j(s_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(s_n) \tag{51}$$

To see this, consider the two cases: $t_n^m(s_n) > 0$ and $t_n^m(s_n) = 0$. In the first case, $\hat{t}_n^m(\hat{s}_n) - t_n^m(\hat{s}_n) = \hat{t}_n^m(s_n) - t_n^m(s_n) = b_n^m$ and (50) implies (51). In the second case,

$$\begin{aligned}
F_n(s_n) + \sum_{j=1}^{m} t_n^j(s_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(s_n) &= F_n(s_n) + \sum_{j=1}^{m-1} t_n^j(s_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(s_n) \\
&\leq \max_{s_n \neq \hat{s}_n} \left(F_n(s_n) + \sum_{j=1}^{m-1} t_n^j(s_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(s_n)\right) \\
&\leq F_n(\hat{s}_n) + \sum_{j=1}^{m-1} t_n^j(\hat{s}_n) + \sum_{j=m}^{N} \hat{t}_n^j(\hat{s}_n) - b_n^m \\
&= F_n(\hat{s}_n) + \sum_{j=1}^{m} t_n^j(\hat{s}_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(\hat{s}_n).
\end{aligned}$$

where the second inequality is due to the definition of $b_n^m$ and the last equality is due to (49). Again, (51) holds.

It is immediate to see that (WT) holds The transfer on $\hat{s}_n$ are always reduced as much as the transfers on the other actions:

$$\hat{t}_n^m(\hat{s}_n) - t_n^m(\hat{s}_n) \geq \hat{t}_n^m(s_n) - t_n^m(s_n) \qquad \forall s_n \in S_n. \tag{52}$$

Finally, to prove (CM), note that for every m, if $b_n^m = \hat{t}_n^m(\hat{s}_n)$, then $t_n^m(\hat{s}_n) = 0$, while, if $b_n^m < \hat{t}_n^m(\hat{s}_n)$, (49) implies

$$t_n^m(\hat{s}_n) = \hat{t}_n^m(\hat{s}_n) - F_n(\hat{s}_n) - \sum_{j=1}^{m-1} t_n^j(\hat{s}_n) - \sum_{j=m}^{N} \hat{t}_n^j(\hat{s}_n) + \max_{s_n \neq \hat{s}_n} \left(F_n(s_n) + \sum_{j=1}^{m-1} t_n^j(s_n) + \sum_{j=m+1}^{N} \hat{t}_n^j(s_n)\right).$$

Then, at least one of the following statements is true: (i) $t_n^m(\hat{s}_n) = 0$; or (ii) there exists an $\tilde{s}_n \neq \hat{s}_n$ such that

$$F_n(\hat{s}_n) + \sum_{j=1}^{m} t_n^m(\hat{s}_n) + \sum_{j=m+1}^{N} \hat{t}_n^m(\hat{s}_n) = F_n(\tilde{s}_n) + \sum_{j=1}^{m-1} t_n^m(\tilde{s}_n) + \sum_{j=m+1}^{N} \hat{t}_n^m(\tilde{s}_n)$$

Also, (CM) can be rewritten as: at least one of the two following statements is true: (a) $t_n^m(\hat{s}_n) = 0$; or (b) there exists an $\bar{s}_n \neq \hat{s}_n$ such that

$$F_n(\hat{s}_n) + \sum_{m \in M} t_n^m(\hat{s}_n) = F_n(\bar{s}_n) + \sum_{j \neq m} t_n^j(\bar{s}_n)$$

If (i), then (a). If (ii), then (52) implies

$$F_n(\hat{s}_n) + \sum_{m \in M} t_n^m(\hat{s}_n) \leq F_n(\tilde{s}_n) + \sum_{j \neq m} t_n^j(\tilde{s}_n)$$

which, combined with (AM), yields (b). ■

With Proposition 11, we can focus on necessary and sufficent conditions for the existence of a vector d that solves (WTd) and (AMd). We use the following duality result:

**Theorem 11** [15] Given a matrix A and a vector a, either (i) there exists an x such that $Ax \leq a$; or (ii) there exists a y such that $yA = 0$, $ya < 0$, and $y \geq 0$.

We rewrite (WTd) and (AMd) in a way that fits (i) of Theorem 11. Let

$$B_{(ms;jna)} = \begin{cases} -1 \text{ if } j = m; s_n = a; s_n \neq \hat{s}_n \\ 0 \text{ otherwise;} \end{cases}$$

$$C_{(ns_n;jia)} = \begin{cases} 1 \text{ if } n = i; s_n = a; \\ 0 \text{ otherwise.} \end{cases}$$

$$b_{ms} = G^m(\hat{s}) - G^m(s) \tag{53}$$

$$c_{ns_n} = F_n(\hat{s}_n) - F_n(s_n) \tag{54}$$

Then B has dimensions $(MS; MH)$, C $(H; MH)$, b $(MS; 1)$, and c $(H; 1)$. If we let $x = d$,

$$A = \begin{bmatrix} B \\ C \end{bmatrix};$$

and

$$a = \begin{bmatrix} b \\ c \end{bmatrix};$$

we transform the problem of the existence of a d satisfying (AMd) and (WTd) into (i) of Theorem 11.

By Theorem 11, (i) is true if and only if there is no y such that (ii) is true. Let $y = [w; z]$, where w has dimensions $(1; M \times S)$ and z has dimensions $(1; H)$. Then (ii) says that $wB + zC = 0$, $wb + zc < 0$, $w; z \geq 0$.

Let $1_{(\cdot)}$ be the indicator function. The system $wB + zC = 0$ can be rewritten as: for every $m \in M; n \in N$ $a_n \in S_n$:

$$-\sum_{j \in M}\sum_{s \in S} w^m(s) 1_{(m=j; s_n \neq \hat{s}_n; s_n = a_n)} + \sum_{i \in N}\sum_{a_n \in S_n} z_n(s_n) 1_{(i=n; s_n=a_n)} = 0;$$

[15] See Mangasarian [12, p. 33].

which we can write

$$\text{for every } m \in M; n \in N; a_n \in S_n/\hat{s}_n : \sum_{\{s: s_n = a_n\}} w^m(s) = z_n(a_n):$$

which, together with the nonnegativity condition $w; z \geq 0$, corresponds to balancedness.

The inequality $wb + zc < 0$ can be transformed into

$$\sum_{m \in M} \sum_{s \in S} w^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n \in N} \sum_{s_n \in S_n} z^n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) < 0;$$

This is the negation of the game being balanced.

We have shown that exactly one of the following two statements is true: the system (WTd) and (AMd) has a solution d, or there exist balanced vectors w and z that violate the balanced game condition. This proves the theorem.

**Proof of Theorem 6** Theorem 5 is stated for a finite S. However, it is easy to see that the proof goes through for an infinite S provided the G's and the F's are bounded and continuous (for a version of Gale's Theorem in a Banach space, see Aubin and Ekeland [2, Corollary 22, p 144]). Rather than introducing the notation for the infinite case, we note that for any collection of balanced weights $\bar{w}$ and $\bar{z}$ that yields

$$\bar{M} = \sum_{m \in M} \sum_{s \in S} \bar{w}^m(s)(G^m(\hat{s}) - G^m(s)) + \sum_{n \in N} \sum_{s_n \in S_n} \bar{z}_n(s_n)(F_n(\hat{s}_n) - F_n(s_n));$$

and for any positive number $\epsilon$, there exists a collection of balanced weights w and z that yields at least $\bar{M} - \epsilon$, but assigns strictly positive weights on only a finite number of elements of S. Hence, if there exists an "infinite" $\bar{w}$ and $\bar{z}$ that violates the condition for a balanced game, then there also exists a "finite" w and z that violates it.

To simplify notation, redefine without loss of generality S and G in a way that $\hat{s} = 0$ and, for all m and n, $G^m(0) = F_n(0) = 0$.

The proof proceeds by contradiction. Suppose there exists no weakly truthful equilibrium with outcome $\hat{s}$. Then, there exists a collection of nonnegative weight w such that, for each agent n, there is a finite set $A_n \subset S_n/\{0\}$ such that

$$\forall m \in M; \forall n \in N; \forall a_n \in A_n : \sum_{\{s: s_n = a_n\}} w^m(s) = z_n(a_n); \tag{55}$$

and, letting $A = \prod_{n \in N} A_n$,

$$\sum_{m \in M} \sum_{s \in A} w^m(s) G^m(s) + \sum_{n \in N} \sum_{s_n \in A_n} z_n(s_n) F_n(s_n) > 0: \tag{56}$$

If necessary, re-scale the weights w and z in a homogeneous way (multiply all of them by the same scalar) so that $\sum_{n \in N} \sum_{s_n \in A_n} z_n(s_n) = 1$. This re-scaling does not unsettle the inequality (56) or the equalities (55), and it implies that

$$\sum_{s \in A} w^m(s) \leq 1 \qquad \forall m \in M;$$

$$\sum_{s_n\in A_n} z_n(s_n) \le 1 \qquad \forall n \in N.$$

Let $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N)$ be defined by $\hat{s}_n = \sum_{s_n\in A_n} z_n(s_n)s_n$ for every n. By (55),

$$\hat{s}_n = \sum_{a_n\in A_n}\left(\sum_{\{s:s_n=a_n\}} w^m(s)\right)a_n = \sum_{a_n\in A_n}\left(\sum_{\{s:s_n=a_n\}} w^m(s)s_n\right) = \sum_{s\in A} w^m(s)s_n.$$

Then, we can write $\hat{s} = \sum_{s\in A} w^m(s)s$ (where the scalar $w^m(s)$ multiplies the vector s). For every m,

$$\begin{aligned} G^m(\hat{s}) &= G^m\left(\sum_{s\in A} w^m(s)s\right) = G^m\left(\sum_{s\in A} w^m(s)s + \left(1-\sum_{s\in A} w^m(s)\right)0\right) \qquad (57)\\ &\ge \sum_{s\in A} w^m(s)G^m(s) + \left(1-\sum_{s\in A} w^m(s)\right)G^m(0) = \sum_{s\in A} w^m(s)G^m(s); \end{aligned}$$

where the inequality is due to the concavity of G. Moreover, by concavity of F:

$$\begin{aligned} F_n(\hat{s}_n) &= F_n\left(\sum_{s_n\in A_n} z_n(s_n)s_n\right) = F_n\left(\sum_{s_n\in A_n} z_n(s_n)s_n + \left(1-\sum_{s_n\in A_n} z_n(s_n)\right)0\right) \qquad (58)\\ &\ge \sum_{s_n\in A_n} z_n(s_n)F_n(s_n) + \left(1-\sum_{s_n\in A_n} z_n(s_n)\right)F_n(0) = \sum_{s_n\in A_n} z_n(s_n)F_n(s_n) \end{aligned}$$

By summing (57) over m and summing (58) over n, and adding the two resulting inequalities, we get

$$\sum_{m\in M} G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) \ge \sum_{m\in M}\sum_{s\in A} w^m(s)G^m(s) + \sum_{n\in N}\sum_{s_n\in A_n} z_n(s_n)F_n(s_n):$$

By (56), this implies $\sum_{m\in M} G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) > 0$, which is a contradiction because $\sum_{m\in M} G^m(0) + \sum_{n\in N} F_n(0) = 0$ was assumed to be the maximum of $\sum_{m\in M} G^m(s) + \sum_{n\in N} F_n(s)$ over s.

**Proof of Theorem 7** In our game, the players are the principals; the strategy space $T^m$ of each principal is the set $[0, K]$. The set $T^*$ is the set $t \in T : \sum_{m\in M} t_n^m(s_n) \neq \sum_{m\in M} t_n^m(\hat{s}_n)$ for every pair of actions $s_n, \hat{s}_n$ and every n; that is, the set of transfers of each principal such that the best choice of action of each player is uniquely defined. This is clearly a dense subset of the space of transfers. Let the function $\phi$ be defined for every $t \in T^*$ as $\phi(t) \equiv G^m(s^*)$ where for each agent $s_n^*$ is the action that maximizes the payoff of the agent. In addition, let the agents use the set of correlated strategies on the actions that give equal payoff, at any vector of transfers of the principals where this occurs. This is the convex completion of the function $\phi$. It is now immediate to check that all the conditions of the general existence theorem of Simon and Zame are satisfied, hence an equilibrium exists.

**Proof of Proposition 8** An efficient equilibrium exists if and only if there exists a pure-strategy equilibrium with outcome $(T; L)$. Let us suppose that such equilibrium exists and that the equilibrium transfers of Principal 1 are given by $\hat{\tau}^1$.

Principal 2 can guarantee herself a gross payoff of $x$ by convincing at least one of the agents to deviate. If Principal 2 offers to Agent 1 $\tau_1^2(BL) > \hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BL)$ and $\tau_1^2(BR) > \hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BR)$, then it is a dominant strategy for Agent 1 to choose $s_1 = B$, independently of what Agent 2 does. Similarly, if Principal 2 offers to Agent 2 $\tau_2^2(TR) > \hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TR)$ and $\tau_2^2(BR) > \hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TL)$, then a deviation of 2 is guaranteed. Hence, in order for $\hat{\tau}^1$ to be an equilibrium transfer, it must be such that

$$\max(\hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BL); \hat{\tau}_1^1(TL) - \hat{\tau}_1^1(BR)) \geq x;$$

$$\max(\hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TR); \hat{\tau}_2^1(TL) - \hat{\tau}_2^1(TL)) \geq x;$$

implying $\hat{\tau}_1^1(TL) \geq x$ and $\hat{\tau}_2^1(TL) \geq x$. Hence, $\hat{\tau}_1^1(TL) + \hat{\tau}_2^1(TL) \geq 2x$, which implies that the net payoff of Principal 1 is negative because, by Assumption, $3 < 2x$. This shows that a pure-strategy equilibrium with outcome $(T; L)$ cannot exist.

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