A Dynamic Equilibrium Model of Firm's Life Cycle and Mergers as Efficient Reallocation*

Kyriakos Frantzeskakis
University of Wisconsin - Madison

Masako Ueda
University of Wisconsin - Madison and CEPR

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Abstract

Any factor that makes acquisition more appealing should increase the number of acquisition that occur. This idea has been captured in standard static models in the literature. However, an increase in the number of acquisitions today means fewer firms exist to perform acquisitions in the future. This dynamic, which we explore, is not well understood. We study a model of mergers motivated by efficient reallocation of projects. A firm may conceive a project that it may not be able to develop successfully. It can be acquired by an established firm that has already proved its ability to develop such projects successfully. We find that, when acquisition costs are low established firms acquire young firms (but not other established firms) in the steady state. More strikingly, if the likelihood of project success decreases for young firms, we find that a higher fraction of young firms attempt to develop their projects rather than to be acquired. This contrasts the previous literature's findings on acquisitions. The explanation for this result is that an increased likelihood of firms' failures causes a shortage of established firms that can then acquire new young firms. Finally, if acquisition costs are moderate we find that established firms acquire other established firms, but not young firms.

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Abstract

Any factor that makes acquisition more appealing should increase the number of acquisition that occur. This idea has been captured in standard static models in the literature. However, an increase in the number of acquisitions today means fewer firms exist to perform acquisitions in the future. This dynamic, which we explore, is not well understood. We study a model of mergers motivated by efficient reallocation of projects. A firm may conceive a project that it may not be able to develop successfully. It can be acquired by an established firm that has already proved its ability to develop such projects successfully. We find that, when acquisition costs are low established firms acquire young firms (but not other established firms) in the steady state. More strikingly, if the likelihood of project success decreases for young firms, we find that a higher fraction of young firms attempt to develop their projects rather than to be acquired. This contrasts the previous literature's findings on acquisitions. The explanation for this result is that an increased likelihood of firms' failures causes a shortage of established firms that can then acquire new young firms. Finally, if acquisition costs are moderate we find that established firms acquire other established firms, but not young firms.

1 Introduction

Young firms often finish their lives by being acquired. For instance, the National Venture Capital Association reports that in 2005, six times as many venture capital (VC) backed firms were acquired as went public. Furthermore, the total value of the acquired firms was at least 4.6 times as much as that of the IPO firms.[1] Brau, Francis, and Kohers (2003) find that in 1998 public firms acquired 1,478 private firms, which is 5.5 times the number of IPOs.

After their IPOs, some firms establish their positions as industry leaders and become active acquirers. For instance, Cisco has acquired 119 companies since 1993, targeting firms with less than 75 employees. (Eisenhardt and Sull, 2001) Microsoft has acquired 117 firms since 1987, Amgen has acquired 22 firms since 1990, and Yahoo has acquired 31 firms since 1997.[2]

As a first step in explaining these observations, we note that a young firm's decision to resist being acquired and to grow today is closely linked to its future opportunities to acquire other young firms. From the perspective of the acquisition market as a whole, today's supply of acquisition targets influences the number of firms that demand acquisition targets in the future. The goal of this paper is to model this dynamic in the acquisition market and study the conditions under which we see more acquisitions of young firms. To this end, we study a dynamic equilibrium model in which mergers are motivated by efficient reallocation of horizontally differentiated projects.

We develop a continuous time model of a stationary economy populated by a continuum of firms. At each point in time, a firm is either *incubating* a project, *developing* a project, or *harvesting* revenues from a project. At the end of incubation, the project's "type" is revealed. If the firm chooses to develop the project, then it succeeds or fails the development stage depending on whether the firm's (initially unknown) "type" matches the project's type. A firm learns its type the first time it develops a project successfully and becomes "established." A successful project generates revenues

[1] http://www.nvca.org/pdf/Exitpollfinalq32006.pdf
[2] Source: SDC Platinum Merger and Acquisitions Database.

throughout a harvesting stage. On the other hand, if the firm chooses not to develop the project, it may either abandon the project entirely or let itself be acquired by another firm. Furthermore, each firm repeatedly faces the decision of whether to begin this project cycle or to proceed directly to the development stage by acquiring another firm that has already incubated a project.

In this model, young firms choose not to acquire other firms but established firms do. Young firms do not know their capabilities and therefore cannot selectively acquire projects that match the firms' capabilities. Nevertheless, once the firms learn their capabilities and become established, they know which projects to acquire. By only acquiring projects that match their capabilities, established firms can avoid failing in development. As a result, acquisitions become more attractive as firms become established.

We find that when acquisition costs are low, established firms always acquire projects from young firm instead of incubating them. Some young firms resist being acquired and develop their projects. The fraction of young firms that choose to be acquired is negatively related both with the length of the project life cycle and with the difficulty of developing a project successfully for the first time. Intuition behind these results is as follows. First, a shorter project life cycle forces the established firms to come back to the acquisition market for new projects more quickly and therefore increases the number of acquisitions. Second, a difficulty of successful development decreases the number of established firms in the future and thereby suppresses the number of acquirers.

It is important to note that our aforementioned findings with regards to the fraction of acquired young firms crucially depend on our assumed inability of each firm to scale up its acquisition capacity. If a firm would be able to costlessly double the number of firms it can acquire by acquiring another firm, such an acquisition would not at all reduce the future demand for acquisitions in the market as a whole. Thus, the dynamic linkage between the number of acquisitions today and the number of acquisitions in the future would disappear and intuition from standard static models of acquisitions would prevail.

We also find that when acquisition costs are moderate, established firms acquire other established firms but not young firms. In this equilibrium some established firms incubate projects hoping that their incubation effort will generate projects that match their own capabilities and that acquisition costs will be saved. Nevertheless, if they incubate projects that do not match their own capabilities, it is optimal to be acquired and to receive payments from the acquirer immediately.

Our paper is related to several strands of literatures. Similarly to this paper, Gans and Stern (2000) model the decision between going public and being acquired as the commercialization strategy of new firms. Hellmann (2006) studies the optimal contract that induces the efficient choice between going public and being acquired. Nevertheless, neither Gans and Stern nor Hellmann studies the acquirer side and the target side simultaneously, whereas we address both sides of the acquisition market. Rhodes-Kropf and Viswanathan (2005) do study both sides of the market and argue that the sufficient condition for acquisitions is not only overvaluation of the acquirers but also overvaluation of the targets. Lambrecht (2004) also studies the decision of an acquirer and a target simultaneously when mergers are motivated by economy of scale. Differently from these two papers, motives for mergers in our model are efficient reallocation of projects. Similarly to our paper, Jovanovic and Braguensky (2004) model mergers motivated by reallocation of projects and study the economy populated by many firms. Nevertheless, their model is static and they therefore do not model any intertemporal linkages in the acquisition market, which our paper does study. Gorton, Kahl, and Rosen (2005) do study the intertemporal linkage, when a firm can only acquire a smaller firm. Differently from their model, we endogenize the direction of acquisition such as an established firm acquiring a younger firm.

In non-merger contexts, a few papers model the learning process of young firms differently from our paper. Jovanovic (1982) is a seminal paper in modeling a new firm's learning process. Nevertheless, projects are homogeneous in the Jovanovic model, and therefore it does not address the issue of project reallocation that is the focus of our paper. Similarly to our paper, Bernardo and Chowdhry (2000) study the firm that learns its capability by experimentation. They assume that a firm may possess either project-specific capability or general-management capability. In equilibrium, an established firm diversifies into a new area to see which type of capability the firm possesses. If the diversification is successful, the firm learns that it possesses the general-management capability. Otherwise, the firm learns that it possesses a project-specific capability. Differently from our paper, the focus of Bernardo and Chowdhry is the firms that have already developed projects successfully. As a result, their model is more appropriate for explaining the learning of established firms rather than that of young firms, which is the focus of our paper. Frantzeskakis and Ueda (2006) employ the same learning model as this paper and study the project selection policy of firms as well as the case where a firm can sell its project without being acquired.

The organization of this paper is as follows. Section 2 describes the basic model. Section 3 characterizes the stationary equilibrium of the model. Section 4 studies the decision of young firms after failing in developing the first project. Section 5 discusses the related evidence. Section 6 concludes.

2 The Model

Our economy is populated by atomless firms and characterized by continuous-time and infinite horizon. Each firm is risk neutral and discounts future cash flows at the continuously compounded discount rate r. Firm's capability is denoted by $s_f \in T_S$, where $T_S = \{1, 2, ..., S\}$. This capability is time-invariant and defines which project the firm can develop profitably. Project's type s_π belongs to the same space T_S as the firm type, and the firm can commercialize a project profitably if and only if the firm's capability and the project's type are the same. We call the firm "young" if it does not yet know its own capability and "established" if it learned its own capability. We assume that the prior probability that a young firm possesses capability s_f is $1/S$, $\forall s_f \in T_S$.

2.1 Project Life Cycle

A project goes through three stages. The first stage is incubation, the second stage is development, and the third stage is harvesting. To incubate a project, a firm has to spend per period cost c_0. The firm completes the incubation according to a negative exponential arrival process with rate of λ_0. When the incubation is completed, the project type $s_\pi \in T_S$ is revealed. Note that the type space of project is the same as that of capability. To develop an incubated project, either the firm that incubated it or another firm has to pay c_1 per period. The firm completes the development according to a negative exponential arrival process with rate of λ_1. If the type of the firm who developed the project coincides with the project's type, the harvesting stage starts once the development is completed. During the harvesting stage, the firm that developed the project receives a per-period profit $y > 0$. The harvesting stage ends according to a negative exponential arrival process with rate of λ_2. We may interpret λ_2 as the obsolescence speed of the project. For instance, if technological innovation is rapid, a rival firm comes up with another project quickly and therefore rent y will be lost quickly.[3] If the capability of the firm who developed the project does not match with the project's type, then the project dies at the end of the development stage and generates no profit.[4]

No matter which stage of life-cycle a firm is in, it may suddenly die. Such a death occurs according to a Poisson process with death rate $\mu > 0$. We denote the present value of the expected incubation cost during the incubation stage by

$$C_0 = c_0 / \left(r + \lambda_0 + \mu\right),$$

the present value of the expected development cost during the development

[3] This interpretation suggests that λ_2 is positively related with incubation and development hazards λ_0 and λ_1. In general, this positive relation reenforces our results.

[4] A more general setting is to assume that y is a decreasing function of the distance between s_π and s_f. This generalization, however, adds a complex learning problem of a young firm. For instance, after generating a sufficiently high revenue, the young firm tries to commercialize if and only if the new project type is sufficiently close to that of the previous project. Studying this learning and experimentation process of the young firm is out of this paper's scope.

stage by

$$C_1 = c_1 / (r + \lambda_1 + \mu),$$

and the present value of the harvesting cash flow during the harvesting stage by

$$Y = y / (r + \lambda_2 + \mu).$$

It is convenient to work with discount factors instead of discount rates, and therefore we introduce the following notation:

$$\rho_j = \frac{\lambda_j}{\lambda_j + r + \mu}, \ j = 0, 1, 2.$$

Discount factor ρ_j is increasing in λ_j because the earlier the current project stage ends, the earlier the firm realizes cash flows in the next stage. Discount factor ρ_j is decreasing in μ because the more likely the firm dies, the less likely it realizes cash flows in the next stage.

2.2 Learning

A young firm may become an established firm by developing a project. If the project of type s starts generating y after the development is completed, the firm learns that its capability s_f is equal to s and therefore becomes established. To highlight the basic tradeoffs of the model, for now, we assume that the young firm exits from the market if the developed project does not generate y. This assumption is reasonable in the presence of financial constraints. A young firm may have a limited financial resource and therefore it may not be able to afford launching another project after failing in one project. In Section 4, we allow a young firm to restart and endogenize the young firm's decision after failing in developing a project.

2.3 Stationary Economy

We now move to describe the entire economy. Let $G_1(t)$ be the mass of young firms incubating at time t, $G_2(t, s)$ be the mass of young firms developing the project s at time t, and $M_i(t, s)$, $i \in \{0, 1, 2\}$ be the mass of established firms with project s in stage i at time t. Let $N(t)$ be the mass of new firms entering in the market at time t and $p(t,s)$ be the price of a project with type s at time t.

We focus on a symmetric stationary economy such that $N(t) = N$, $G_0(t) = G_0$, $G_1(t) = G_1$ for $\forall t$, $p(t,s) = p$, for $\forall t, s$, and $M_j(t,s) = M_j/S$ for $j \in \{0,1,2\}$ and $\forall t, s$. We assume that $N > 0$. Otherwise, the only stationary equilibrium is that N, G_0, G_1, M_0, M_1, and M_2 are all zero. This is because due to $\mu > 0$, it is necessary to have a positive inflow into each mass in order to sustain constant masses.

2.4 Value of the Firm

We denote each stage of firm's life-cycle by $\sigma \in \{g/0, g/1, m/0, m/1, m/2\}$, where g means that a firm is young and m means that a firm is established. The symbol /0 means that the firm is incubating a project, /1 means that the firm is developing a project, and /2 means that the firm is generating revenues. Using this notation, we now define the value of the firm for each stage of the firm's life cycle. Let $i = \{g, m\}$ be the firm's age and $j = \{0,1,2\}$ be the project-related stage of the firm as defined above. We denote by $V_{i/j}$ the value of the firm whose age is i and whose product stage is j. Value $V_{i/j}$ is the discounted future expected cash flow of the firm given that its future decision is optimal. Since only established firms harvest, we denote $V_{m/2}$ by V_2 in what follows.

Value of the firms which operate in equilibrium is subject to two types of selection bias. First, we normalize such that if the firm ceases to operate or gets acquired, then its value becomes zero. Therefore, values of any firms choosing to operate in equilibrium should be non-negative. Second, in equilibrium, an established firm should only develop a project that matches the firm's capability and should avoid developing an unmatched project. Therefore, $V_{m/1}$ is the value of an established firm conditioned on that the project being developed matches with the firm's capability.

2.5 Acquisitions

When a firm completes incubating a project, it may sell the project to another firm. We assume that this project sales occurs as the latter firm's acquisition of the former firm. In other words, we do not allow the seller firm to separate itself from its project. Such separation may not be possible, when the knowledge generated through the incubation process is tacit and sticky to the incubator firm, as emphasized by Teece (1988). The consequence of this assumption is that once the incubator firm decides to sell its project, the firm cannot recycle its incubation ability for another new project. The rarity of research specialized firms corroborates the relevance of our assumption. Arora, Fosfuri, and Gambardella (2001) summarize the intellectual history about this rarity as follows. "Stigler (1951) argued that division of labor could also embrace the innovation process and industry evolution would lead to the rise of stand-alone R&D labs selling their research outcomes to other parties. Thus, far, this prediction had not come true. Mowery (1983) showed that employment of scientific personnel in independent research organizations dropped between the two wars. More generally, the historical evidence suggests that since the nineteenth century, manufacturing companies have increasingly internalized R&D operations (Chandler 1990)."

We model acquisition as follows. A firm pays p to a seller firm and acquires the project incubated by the seller firm. At this transaction, the seller firm incurs dissipative transaction cost equal to $\psi > 0$. One interpretation of this transaction cost is legal costs for the seller firm to secure its intellectual property to its project before contacting the acquirer. In order for the seller firm to convince the acquirer to buy the seller firm, it presumably needs to disclose some of its intellectual property. Without establishing its own rights to such intellectual property in advance, the potential acquirer may be able to steal the seller's intellectual property. Another interpretation of this transaction cost is search cost. When a firm decides to be acquired, it has to find an acquirer. Finding an acquirer takes not only time and effort of the firm but also fees the firm has to pay to brokers including investment bankers.

In modeling the acquisition process described above, we assume that a firm with a suitable development capability acquires an incubator firm, *not the other way around*. We find a justification for this assumption in the literature on property rights approach such as Grossman and Hart (1986), which emphasizes the role of property rights in inducing relation-specific investments. At a development stage of the project, investments by the incubator firm is sunk, whereas the developing firm still needs to make investments specific for developing the incubated project. As such investments are likely to be not contractible, it is important to give an adequate incentive to the developing firm at this stage. Owning the assets of the incubator firm, the developing firm can strengthen its own ex-post bargaining power and therefore becomes more willing to make relation-specific investments. Nevertheless, if the incubator firm acquires the developing firm, the latter's incentive is undermined. As a result, it is more efficient for the developing firm to acquire the incubator firm than the other way around.

2.6 Life-cycle of Firms

A young firm starts its life with a decision between acquiring an incubated project and incubating a new project. If it decides to acquire the incubated project, its value is equal to $V_{g/1} - p$, and $V_{g/0}$, otherwise. Its value is $V_{g/0}$ during the incubation. Once it completes the incubation, it chooses either developing the incubated project by itself or being acquired. If it decides to be acquired, it gets $p - \psi$. If it decides to develop the project, then its value becomes $V_{g/1}$. Thus,

$$V_{g/0} = \rho_0 \max\left\{p - \psi, V_{g/1}\right\} - C_0. \tag{1}$$

When the development ends, the project matches with the firm's capability with probability $1/S$. In this case, the firm moves to the harvesting stage and gets V_2. With the probability $(S-1)/S$, the project does not match with the firm's capability. In this case, at the completion of the development stage, the firm exits and gets zero. Therefore,

$$V_{g/1} = \frac{\rho_1 V_2}{S} - C_1 \tag{2}$$

Once the harvesting period ends, the established firm chooses either incubating a new project and getting $V_{m/0}$ or acquiring the project that fits its own capability by paying p. If the latter is chosen, the firm proceeds to develop the acquired project and its value becomes $V_{m/1}$. Thus,

$$V_2 = Y + \rho_2 \max \left\{ V_{m/0}, V_{m/1} - p \right\}. \tag{3}$$

We now describe an incubation process of an established firm. Once getting to know its own type, the firm may naturally try to avoid incubating a project that does not fit the firm's capability. We model this idea of directed incubation as follows. Suppose that an established firm with capability s_f chooses to incubate a new project. Then, a new project of type s, $\forall s \neq s_f$ will arrive with probability $\kappa \leq S^{-1}$. Because there are $S-1$ types of such projects, an established firm incubates an unmatched project with probability $(S-1)\kappa$ and a project of type s_f with probability $1-(S-1)\kappa$. Note that if $\kappa = S^{-1}$, the incubation process of established firms is the same as that of young firms.

If the incubated project turns out to fit the established firm's capability, the firm proceeds to develop the project and the firm's value becomes $V_{m/1}$. Otherwise, the firm chooses between being acquired, incubating another project, or acquiring an incubated project. Therefore,

$$\begin{aligned} & V_{m/0} \\ = \; & (1-(S-1)\kappa)\,\rho_0 V_{m/1} + (S-1)\,\kappa\rho_0 \max\left\{p-\psi, V_{m/0}, V_{m/1}-p\right\} - C_0 \end{aligned} \tag{4}$$

Once the established firm starts developing a project that matches with the firm's capability, the firm just waits for the harvesting period while incurring the development cost. Thus,

$$V_{m/1} = \rho_1 V_2 - C_1. \tag{5}$$

It is worthwhile to mention that the life-cycle described above is a reduced form in the following sense. An established firm naturally has an option to sell its project irrespective of the project type. But we ignore the possibility that the firm may sell the project when it matches with the firm's capability. This is because in equilibrium the firm that benefits most from possessing a certain type of a project is the one with the matching capability. Besides, selling a project costs ψ. Therefore, the equilibrium price of the project should not be so high that even the firm with the matching capability is willing to sell it to another firm.

It is useful to discuss three issues on our basic setup at this moment. First, the order of the learning, project type first and capability later, is important in our model. If the firm would learn its capability first, we would not have the difference between a young firm and an established firm that we derive later. Nevertheless, we can potentially generalize the learning process such that a firm may learn of its capability prior to learning of project type, as far as the firm may not learn of its capability precisely and therefore there is still value of learning from the development outcome. Second, "firm" characterized by s_f may be a collection of resources such as entrepreneur, partners, knowledge, physical assets including location, or unique products or services. Nevertheless, our "firm" needs to be a distinct entity from its project. Third, we assume that each firm can undertake only one project at a time. This is a convenient way to model a firm whose production is subject to decreasing returns to scale.

3 Analysis of Stationary Equilibrium

Given our setup presented in the last section, we are now going to characterize equilibrium. Note that we need a positive mass of young firms turning into established firms, because otherwise the mass of established firms shrinks due to a positive death rate $\mu > 0$. Thus, when a young firm finishes incubating a project, it should at least weakly prefer developing the project to being acquired. Therefore,

$$p - \psi \leq V_{g/1}. \tag{6}$$

Using this inequality, we can replace equation (1) with the following.

$$V_{g/0} = \rho_0 V_{g/1} - C_0. \tag{7}$$

We are now going to examine who acquires whom in equilibrium. We begin with showing that a young firm does not acquire any firm. We then proceed to characterize the equilibrium in which an established firm acquires another firm.

3.1 Non-Existence of Equilibrium with Mergers between Young Firms

We are now going to show that a young firm does not acquire any firm in equilibrium. First, we will show that a young firm should not acquire another young firm and second, we will show that a young firm should not acquire an established firm.

For mergers between two young firms to occur, a young firm needs to be willing to acquire another young firm, as well as another young firm is willing to incubate a project and then be acquired. The first condition is summarized to

$$V_{g/1} - p \geq V_{g/0}. \tag{8}$$

That is, a young firm prefers acquiring an incubated project instead of incubating its own. The second condition requires the following two conditions:

$$V_{g/1} - p \leq V_{g/0} \tag{9}$$

and

$$p - \psi \geq V_{g/1}. \tag{10}$$

Proposition 1 *There does not exist equilibrium in which a young firm acquires another young firm.*

Proof. See Appendix.

Intuition behind this proposition is straightforward. If a young firm acquires another young firm, such an acquisition does not change the probability that the project reaches the harvesting stage. Nevertheless, such an acquisition creates transaction cost ψ as well as causes the loss of the acquired firm from the economy. Therefore, it is inefficient and a gain from trade does not exist.

For a young firm's acquisition of an established firm to occur, two conditions are needed. First, a young firm prefers acquiring an incubated project to incubating its own project. That is, equation (8) needs to hold. Second, an established firm prefers incubating an own project to acquiring an incubated project. Thus,

$$V_{m/0} \geq V_{m/1} - p. \tag{11}$$

The next proposition states that these two conditions are not compatible.

Proposition 2 *There does not exist equilibrium in which a young firm acquires another established firm.*

Proof. See Appendix.

Intuition behind this proposition is as follows. A gain from acquiring an incubated project instead of incubating a new project is for the acquiring firm to save the incubation cost and to move to the development stage without delays. This type of gains exists no matter which a young or established firm is acquiring. In addition, if the acquirer is an established firm, it can selectively acquire the incubated project that matches its own capability and cam therefore avoid the risk of incubating an unmatched project. As a consequence of this additional gain, acquiring an incubated project always brings a bigger benefit to established firms than young firms. Thus, if a young firm would be acquiring, so should be an established firm and no established firm should incubate a project of its own. As a consequence, a young firm should not acquire an established firm in equilibrium.

3.2 Equilibrium in which an Established Firm Acquires a Young Firm

We now construct an equilibrium in which an established firm acquires a young firm. In this equilibrium, a young firm that finishes incubating a project should at least weakly prefer being acquired to developing the incubated project. Thus, $V_{g/1} \leq p - \psi$. Combining this fact with equation (6), we have

$$p = \psi + V_{g/1}. \tag{12}$$

Note that in this equilibrium, a young firm does not have any incentive to acquire another firm because $V_{g/1} - p = -\psi < 0$.

There are six unknown variables $V_{g/0}, V_{g/1}, V_2, V_{m/0}, V_{m/1}$ and p and six linearly independent equations (2), (3), (4), (5), (7), and (12). Therefore, this system has a unique solution. For the solution to be compatible with the established firm's incentive to acquire another firm, an established firm should at least weakly prefer acquiring to incubating its own project. Thus,

$$V_{m/0} \leq V_{m/1} - p. \tag{13}$$

3.2.1 Existence of Equilibrium

We begin with examining if there exists an equilibrium in which established firms specialize in developing projects and incubate no projects. In such an equilibrium, $M_0 = 0$. The following proposition summarizes the results.

Proposition 3 *If*

$$\frac{\rho_0 \rho_1 (Y - \rho_2 \psi)}{S(1 - \rho_1 \rho_2) + \rho_1 \rho_2} - \rho_0 C_1 - C_0 \geq 0 \tag{14}$$

and either (i)

$$\frac{(S-2)\rho_1 Y - (S - \rho_1 \rho_2)\psi}{S(1 - \rho_1 \rho_2) + \rho_1 \rho_2} + C_1 \geq 0 \tag{15}$$

and

$$\begin{aligned} & \frac{(1-\rho_0) S - (1 - (S-1)\kappa \rho_0)}{S(1 - \rho_1 \rho_2) + \rho_1 \rho_2} \rho_1 Y + (1 - (S-1)\kappa) \rho_0 C_1 + C_0 \\ \geq\ & \frac{S - S\rho_0 \rho_1 \rho_2 + (S-1)^2 \kappa \rho_0 \rho_1 \rho_2 - (S - \rho_1 \rho_2)(S-1)\kappa \rho_0}{S(1 - \rho_1 \rho_2) + \rho_1 \rho_2} \psi, \end{aligned} \tag{16}$$

or (ii)

$$\frac{(S-2)\rho_1 Y - (S - \rho_1 \rho_2)\psi}{S(1 - \rho_1 \rho_2) + \rho_1 \rho_2} + C_1 < 0 \tag{17}$$

and

$$\frac{(S-1)\left(1+(S-1)\kappa\rho_0\right)-S\rho_0}{S\left(1-\rho_1\rho_2\right)+\rho_1\rho_2}\rho_1 Y+\rho_0 C_1+C_0$$
$$\geq \frac{S-S\rho_0\rho_1\rho_2+(S-1)^2\kappa\rho_0\rho_1\rho_2}{S\left(1-\rho_1\rho_2\right)+\rho_1\rho_2}\psi, \tag{18}$$

then there exists an equilibrium such that $M_1, M_2 > 0$. In this equilibrium, the project price is given by

$$p=\frac{S\left(1-\rho_1\rho_2\right)\psi+\rho_1 Y}{S\left(1-\rho_1\rho_2\right)+\rho_1\rho_2}-C_1. \tag{19}$$

Proof. See Appendix.

The equation (14) ensures that the value of a new incubating firm is nonnegative such that entry is rational. Not surprisingly, this condition is likely to be satisfied if revenue Y is high, incubation development, and acquisition costs, C_0, C_1 and ψ are low, and it is easier to develop a project successfully for the first time (low S).

To make sure that an established firm is willing to acquire an incubated project instead of incubating its own, we have to check the established firm's payoff from incubating. There are two cases depending on what an established firm wants to do if it ends up incubating an unmatched project. In the first case, the acquisition price is low and therefore the firm wants to acquire an incubated project. In the second case, the acquisition price is high and therefore the firm wants to be acquired. Equation (15) warrants the first case and equation (17) warrants the second case. For each of these two cases, equations (16) and (18) respectively describe the condition under which an established firm prefers acquiring to incubating.

A few things are common for equations (16) and (18). First, they are both more likely to be satisfied if Y is high, because if so, an established firm benefits more from realizing Y earlier by acquiring an incubated project. Second, if C_1 is high and/or ψ is low, the two equations are more likely to be satisfied. This relation occurs because a higher C_1 and/or lower ψ encourages a young firm to be acquired instead of developing its project. As a consequence, the acquisition price p is suppressed as stated in equation

(19). Therefore, an established firm is more likely to acquire instead of incubating. Third, if C_0 is high, the two equations are more likely to be satisfied, because if so, an established firm is less willing to incubate and more inclined to acquire an incubated project.

3.2.2 Distribution of Firms

The stationary equilibrium is characterized by the following system of equations.

$$(\mu + \lambda_0)\, G_0 = N, \tag{20}$$

$$(\mu + \lambda_1)\, G_1 = \lambda_0 G_0 \left(1 - H_g\right), \tag{21}$$

$$(\mu + \lambda_1)\, M_1 = \lambda_2 M_2, \tag{22}$$

$$(\mu + \lambda_2)\, M_2 = \frac{\lambda_1 G_1}{S} + \lambda_1 M_1, \tag{23}$$

$$M_0 = 0 \tag{24}$$

and

$$\lambda_2 M_2 = \lambda_0 G_0 H_g, \tag{25}$$

where H_g is the proportion of projects incubated by young firms and then acquired by established firms. Equation (20) ensures the outflow and the inflow of incubating new firms are the same. Note that among the young firms that have just finished their incubations, the fraction of H_g is sold to established firms and the rest will develop their projects by their own. Then, equation (21) ensures that the outflow and the inflow of developing young firms are the same. Note that the right hand side of equation (22) is the inflow of developing established firms. Then, equation (22) ensures that the outflow and the inflow of developing established firms are the same. Equation (23) ensures that the outflow and the inflow of harvesting firms are the same. Equation (24) ensures that there is no incubating established firm. Finally, equation (25) defines the equilibrium condition in the acquisition market such that the demand for acquisition is equal to the supply.

Given the death rate μ, the entry mass N, and the rates to move to the next stage, λ_0, λ_1 and λ_2, one can solve for the complete distribution G_0, G_1, M_1 and M_2. In what follows, we characterize this stationary equilibrium.

Proposition 4 *In equilibrium in which an established firm acquires a young firm, the rate of acquisition defined by H_g is decreasing in μ and S and increasing in λ_1 and λ_2.*

Proof. See Appendix.

Intuition behind this proposition is as follows. When μ and S are high, fewer firms become established and, as a result, the established firms' demand for projects is low and fewer acquisitions occur. The positive relation between S and the acquisition rate is paradoxical. If S is high, development attempts by young firms are more likely to fail. Nevertheless, the frequency of such an attempt is high if S is high. When λ_1 and λ_2 are high, established firms come back to the project market quickly and, as a result, the demand for projects increases. As a consequence, the acquisition rate also increases.

Proposition 5 *In equilibrium where an established firm acquires a young firm, the values of established firms V_2 and $V_{m/1}$ are both increasing in S. The values of young firms $V_{g/0}$ and $V_{g/1}$ are both decreasing in S.*

Proof. See Appendix.

Intuition behind this proposition is similar to how entry deterrence asymmetrically influences incumbents and new entrants in a basic model of industrial organization. When S is high, it is unlikely that a young firm can successfully develop the project it has just incubated. Developing a project is an outside option of the young firm when it negotiates the price of selling the project to an established firm. If this outside option is not attractive, the bargaining power of the young firm falls and so does the price of the project. Opposite to young firms, an established firm can buy a project at a cheaper price if S is high, and therefore the value of the established firm rises.

3.3 Equilibrium with Mergers between Established Firms

We are now going to examine if there exists a stationary equilibrium in which some established firms incubate and some established firms merge.

3.3.1 Existence of Equilibrium

We begin our analysis by describing the necessary conditions for a stationary equilibrium with mergers between established firms to exist. The first condition is that an established firm needs to be willing to incubate instead of buying an already incubated project, that is, $V_{m/0} \geq V_{m/1} - p$. The second condition is that an established firm is also willing to buy an incubated project instead of incubating its own, that is, $V_{m/1} - p \geq V_{m/0}$. Therefore,

$$p = V_{m/1} - V_{m/0}. \tag{26}$$

In addition, an established firm that has just incubated an unmatched project should be willing to sell it, that is, $\max\left\{V_{m/1} - p, V_{m/0}\right\} \leq p - \psi$. Combining this relation with equation (6), we have

$$\max\left\{V_{m/1} - p, V_{m/0}\right\} \leq p - \psi \leq V_{g/1}. \tag{27}$$

Similarly to the case in which an established firm acquires a young firm, there are six unknown variables $V_{g/0}, V_{g/1}, V_2, V_{m/0}, V_{m/1}$ and p and six linearly independent equations (2), (3), (4), (5), (7), and (26). Therefore, these unknown variables have a unique solution. The following proposition characterizes the stationary equilibrium with mergers between established firms.

Proposition 6 *If*

$$\frac{(1 + (S-1)\,\kappa\rho_0)\,Y - (S-1)\,\kappa\rho_0\rho_2\psi}{S\,(1 - \rho_0\rho_1\rho_2 + (S-1)\,\kappa\rho_0) + \rho_0\rho_1\rho_2}\rho_0\rho_1 - \rho_0 C_1 - C_0 \geq 0, \tag{28}$$

$$\begin{aligned} & (1 - \rho_0\rho_1\rho_2 - (S-1)\,\kappa\rho_0\,(1 - \rho_1\rho_2))\,\psi \\ \leq\ & (1 - 2\rho_0 + (S-1)\,\kappa\rho_0)\,(\rho_1 Y - C_1) + (2 - \rho_1\rho_2)\,C_0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} & \frac{(S-1)\,(1 + \rho_0\,(S-1)\,\kappa) - S\rho_0}{S\,(1 - \rho_1\rho_2) + \rho_1\rho_2}\rho_1 Y + \rho_0 C_1 + C_0 \\ \leq\ & \frac{S - \rho_0\rho_1\rho_2 S + (S-1)^2\,\kappa\rho_0\rho_1\rho_2}{S\,(1 - \rho_1\rho_2) + \rho_1\rho_2}\psi, \end{aligned} \tag{30}$$

then there exists a stationary equilibrium in which established firms merge. The equilibrium price of the project is

$$p = \frac{(1-\rho_0+(S-1)\,\kappa\rho_0)\,(\rho_1 Y - C_1) + (1-\rho_1\rho_2)\,(C_0+(S-1)\,\kappa\rho_0\psi)}{1-\rho_0\rho_1\rho_2+(S-1)\,\kappa\rho_0}. \tag{31}$$

Proof. See Appendix.

Equation (28) guarantees that entry is profitable. Equation (29) warrants that an established firm is willing to be acquired when it incubates an unmatched project. This condition is more likely to be satisfied when ψ is lower and/or C_0 is higher, because being acquired becomes more attractive relative to incubating another project.

Equation (30) is more likely to be satisfied when C_0 and C_1 are low. When C_0 is low, incubation becomes attractive relative to acquiring an incubated project. Low C_1 increases the acquisition price and thereby also makes incubation attractive relative to acquisition. Equation (30) is more likely to be satisfied when ψ is high. This relation is obtained because a high ψ makes an acquisition more costly and therefore an established firm is more willing to incubate. This established firm's willingness to incubate is a necessary condition for mergers between two established firms to occur.

Similarly to the results in the equilibrium in which an established firm acquires a young firm, the project price is increasing in both the revenue and the transfer cost and decreasing in the development cost. Differently from the previous results, the project price DOES depend on the incubation cost and their relation is positive. In an equilibrium with mergers between two established firms, an established firm needs to be indifferent between acquiring and incubating. If the incubation cost is high, the project price must be also high so that the two options continue to be equally attractive.

The following corollary summarizes the necessary condition for existence of the two different types of stationary equilibrium we have discussed.

Corollary 1 *Suppose that there exists an equilibrium in which an established firm acquires another firm. If equation (18) is satisfied, then an es-*

tablished firm acquires a young firm. Otherwise, an established firm acquires another established firm.

Proof. Adding equation (15) times $(1-(S-1)\kappa)\rho_0$ to (16) gives equation (18). Therefore, if an established firm acquires a young firm in equilibrium, then equation (18) must hold. Nevertheless, it is not compatible with equation (30), which is a necessary conditions for existence of equilibrium in which an established firm acquires another established firm. Q.E.D.

This corollary implies that there does not exist multiple equilibria given exogenous variables fixed. In particular, if the transaction cost of acquisition is low, established firms acquire young firms in equilibrium. If the transaction cost of acquisition is moderate, then established firms acquire other established firms in equilibrium. if transaction cost of acquisition is high, no acquisition occurs in equilibrium.

3.3.2 Distribution of Firms

The stationary equilibrium is characterized by the following system of equations.

$$(\mu+\lambda_0)\,G_0 = N, \tag{32}$$

$$(\mu+\lambda_1)\,G_1 = \lambda_0 G_0, \tag{33}$$

$$(\mu+\lambda_1)\,M_1 = (1-(S-1)\,\kappa)\,\lambda_0 M_0 + H_m\lambda_2 M_2, \tag{34}$$

$$(\mu+\lambda_2)\,M_2 = \frac{\lambda_1 G_1}{S} + \lambda_1 M_1, \tag{35}$$

$$(\mu+\lambda_0)\,M_0 = (1-H_m)\,\lambda_2 M_2, \tag{36}$$

and

$$H_m\lambda_2 M_2 = (S-1)\,\kappa\lambda_0 M_0. \tag{37}$$

Similarly to the case in which an established firm acquires a young firm, equation (32) ensures the outflow and the inflow of incubating new firms are the same. Equation (33) ensures that the outflow and the inflow of developing young firms are the same. Note that among the established firms that have just finished their harvesting, the fraction of H_m acquire other established firms and directly move to the development stage. Established firms that have just incubated the projects that match their capabilities also move to the development stage. Then, equation (34) ensures that the outflow and the inflow of developing established firms are the same. Equation (35) and (36) ensure that the outflow and the inflow of harvesting firms and incubating established firms are the same, respectively. Finally, equation (37) guarantees that the demand for acquisition is equal to the supply.

The following proposition characterizes the equilibrium distribution of firms.

Proposition 7 *The proportion of established firms that acquire incubated projects relative to those incubate own projects, H_m, is increasing in κ and λ_0 and decreasing in μ.*

Proof. See Appendix.

Intuition behind this proposition is as follows. First, H_m is increasing in κ because if κ is higher, the incubated project is less likely to match the incubator's capability. As a result, more projects are sold to the firms with the right capabilities for the projects. Second, H_m is increasing in λ_0 because faster incubation translates into a bigger supply of incubated projects. Third, H_m is decreasing in μ because if μ is high, then incubating established firms are more likely to die and less likely to supply incubated projects.

4 Sequential Learning

So far we have assumed that a new firm exogenously exits if it fails in its development attempt. In this section, we allow the firm to use the knowledge obtained from its past development failures. This knowledge is helpful in two respects.

First, if the firm has failed in developing a project of a certain type, then the firm can reduce the chance of incubating another project of the same type. We model this directed incubation effort similarly to the previous sections. Let $s(n)$, $n = 1, 2, ..., S - 1$ be the type of the nth project that the firm's development attempt has failed. Suppose that the firm has failed in developing exactly N types of projects for $N = 1, ..., S - 2$. Then, for the $(N+1)$th project, the firm incubates a project of type $s_p \neq s(n)$ with probability $(1 - N\kappa)/(S - N)$ and $s_p = s(n)$ with probability $0 \leq \kappa \leq 1/S$ for $n = 1, ..., N$.

Let $V_{g/i}[N]$, $i = 0, 1$ be the value of a young firm that has failed in developing exactly N different types of the projects in the past. Similarly to the previous sections, "$i = 0$" and "$i = 1$" denote that the firm is incubating and developing respectively. As the number of project types that potentially matches with the firm's capability is $S - N$, the firm incubates such projects with probability $(1 - N\kappa)$ and proceeds to either develop it or be acquired. (Note that developing the project dominates the option to incubate a new project or acquiring another project.) As the number of project types that, the firm already knows, do not match the firm's capability is N, the firm incubates a project of the same type as one of those the firm failed in the past with probability $N\kappa$. In this case, the firm may either incubate another project, acquire a new project, or be acquired. Then, for $N = 0, 1, ..., S - 2$,

$$\begin{aligned} V_{g/0}[N] &= \rho_0 (1 - N\kappa) \max\left\{V_{g/1}[N], p - \psi, 0\right\} \\ &\quad + \rho_0 N\kappa \max\left\{V_{g/0}[N], V_{g/1}[N] - p, p - \psi, 0\right\} - C_0. \quad (38) \end{aligned}$$

Note that

$$V_{g/0}[S-1] = V_{m/0} \quad (39)$$

because once a firm failed in commercializing $S - 1$ different types of projects, the one type of project that the firm never failed should fit the firm's capability and therefore the firm becomes established.

Second, development failures in the past also helps the firm with developing a project selectively. If the firm has failed in developing a project of a certain type, then the firm can avoid developing another project of the same type and wasting the development cost. In particular, if the young firm has failed in developing exactly N different types of projects, then the firm only develops one of the other $S - N$ types of projects. Conditioned on this pre-screening, the value of a developing young firm is for $N = 0, 1, ..., S-2$

$$\begin{aligned} V_{g/1}[N] &= \frac{1}{S-N}\rho_1 V_2 \\ &+ \frac{S-N-1}{S-N}\rho_1 \max\left\{V_{g/0}[N+1], V_{g/1}[N+1]-p, 0\right\} - \psi \end{aligned} \tag{40}$$

and

$$V_{g/1}[S-1] = V_{m/1}. \tag{41}$$

The value $V_{g/1}[N]$, $N = 0, 1, ..., S-2$ consists of the value obtained if the developed project matches the firm's capability and the value obtained otherwise, less the development cost. Even if the development fails, the firm learns that the type of the project the firm just failed in developing does not match the firm's capability. With this new knowledge, the firm decides between incubating another project to get $V_{g/0}[N+1]$ or acquiring an incubated project to get $V_{g/1}[N+1] - p$.

Proposition 8 *Suppose that in equilibrium*

$$0 \leq p - \psi \leq V_{g/1}[0],$$

$$V_{g/0}[0] \geq 0$$

and

$$V_{m/1} - p \geq V_{m/0}.$$

Then,

$$V_{g/i}[N-1] < V_{g/i}[N] \quad N = 0, 1, ..., S-1 \textit{ and } i = 0, 1.$$

Proof. See Appendix.

Intuition behind this proposition is straightforward. Failing in developing many types of projects helps the firm to narrow down the set of possible capabilities it has. Therefore, a firm's value always increases with the number of development failures in the past.

This proposition helps characterize the two types of equilibria we discussed in the previous section when a young firm is allowed to reenter the market after failing in development. In the equilibria we discussed in the previous section, $V_{g/1}[0] \geq p - \psi$. Due to Proposition 8, $V_{g/1}[N] > V_{g/1}[0]$ for $N = 1, ..., S-1$. Therefore, in these equilibria, the firm with past failures in development strictly prefers developing a project that may match the firm's capability to being acquired. Nevertheless, the firm may prefer to be acquired if it incubates a project that, the firm knows, does not match its capability. In particular, in the equilibrium where an established firm acquires another established firm, $\max\left\{V_{m/1} - p, V_{m/0}\right\} \leq p - \psi$ as discussed in the previous section. Therefore, Proposition 8 implies that the young firm with past failures in development always chooses to be acquired if it incubates a project that, the firm knows, does not match its capability, since $\max\left\{V_{g/1}[N] - p, V_{g/0}[N]\right\} \leq \max\left\{V_{m/1} - p, V_{m/0}\right\}$.

5 Related Evidence

Our model implies that an acquired project has a greater life expectancy and also a higher profitability than a non-acquired project, because an acquired project is developed by the firm with the matching capability, whereas a non-acquired project may not be. Consistent with this prediction, Ravenscraft and Scherer (1987) find that acquired lines of business are of higher quality (usually measured by profitability) than those not acquired. If we interpret an IPO as the young firm's decision to resist being acquired and to develop its project, then our model predicts that a newly public firm performs worse than an acquired firm. Fama and French (2004) report that out of 438 total IPOs between 1990 and 2000, only 225 survived for five years. Jain and Kini (1994) find that IPO firms' operating performance is worse than the matching firms' performance.

Our model also helps predict under what conditions a higher fraction of young firms choose to be acquired instead of going public. Brau, Francis, and Kohers (2003) find that takeovers are more often relative to IPOs in manufacturing industries than in other industries. Differences in acquisition costs between manufacturing industries and others may drive this pattern. Intellectual properties in manufacturing industries are embodied in physical products, whereas in other industries such as service, intellectual properties are embodied in the firm's ways of doing businesses. Property rights to the latter are less easy to establish than the former. Therefore, when a manufacturing firm decides to be acquired, it may incur a lower legal cost than a non-manufacturing firm. According to Corollary 1, when acquisition costs are low established firms should acquire young firms instead of other established firms.

Our model characterizes some commonly observed patterns of acquisitions. First, our model predicts that an established firm acquires a young firm, but not the other way around. Although evidence documenting this age relation between an acquirer and a target is sparse, a plenty of evidence suggests that an acquirer tends to be bigger than its target. Given that firm size and firm age are positively related, the evidence on firm size gives an indirect support for our prediction. Based on the data on all U.S. publicly traded firms that announced M&As between 1994 and 2000, Mitchell, Pulvino, and Stafford (2004) find that the ratio of the median target equity value to acquirer equity value varies between 0.17 (in 1994) and 0.10 (in 2000). That is, the size of an acquirer is usually 6-10 times as much as that of its target, supporting our theoretical prediction. Second, our model predicts that when an established firm has lost a cash cow project, it will initiate acquisitions. Consistent with our model, Higgins and Rodriguez (2006) find that pharmaceutical firms that are more desperate for blockbuster drugs are more likely to engage in outsourcing-type acquisitions.

6 Conclusion

This paper studies a dynamic of the acquisition market when each firm's gain from acquiring another firm endogenously changes over the firm's life cycle. When the firm is young, it does not know its capability and therefore cannot selectively acquire a project that matches the firm's capability. If the young firm happens to develop a project that matches the firm's capability, the firm learns its capability and becomes established. When the firm is established, it can selectively acquire a project that matches the firm's capability—therefore it starts acquiring other firms. We find, in a stationary equilibrium, that when acquisition costs are low, established firms acquire young firms, and that when acquisition costs are moderate, established firms acquire other established firms. In the former equilibrium, young firms that develop their projects and those acquired coexist. When the young firms are more likely to develop their projects successfully, the fraction of young firms that choose to do so decreases and the fraction of those acquired increases. This paradoxical result originates from the dynamic of the acquisition market. A factor that discourages young firms to be acquired and that motivates them to develop their projects today increases the number of established firms who want to acquire other firms tomorrow. As a consequence, the factor that discourages acquisitions in the short-run may stimulate acquisitions in the long-run.

To highlight this dynamic of the acquisition market, we employ a simple model of a firm that can undertake only one project at a time. A merit of this assumption is, first, that it simplifies the model and, second, that it is the simplest form of the well-accepted assumptions in economics that production at firm level is subject to decreasing returns to scale. Nevertheless, we believe relaxing this one-project-at-a-time assumption is an interesting extension. For instance, if we allow the firm to undertake more than one project, our model can potentially address the issues such as firm's growth through acquisition and diversifying acquisitions.

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Appendix

This appendix gathers the proofs of the propositions.

Proof of Proposition 1

We show that equations (8), (9), and (10) are not compatible. Since $V_{g/0}$ and $V_{g/1}$ are non-negative and ψ is positive,

$$\begin{aligned} V_{g/0} + V_{g/1} &> V_{g/1} - \psi \\ &= \left(V_{g/1} - p\right) + (p - \psi)\,. \end{aligned}$$

This contradicts to equation (8) and (10). Therefore, an equilibrium in which a young firm acquires another young firm does not exist. Q.E.D.

Proof of Proposition 2

Suppose that equation (8) and (11) hold simultaneously. Then,

$$V_{g/1} - V_{g/0} \geq V_{m/1} - V_{m/0}. \tag{42}$$

If $p - \psi \geq \max\left\{V_{m/0}, V_{m/1} - p,\right\}$, that is, an established firm is willing to be acquired, then substituting the definition of $V_{g/0}$ and $V_{m/0}$ into equation (42) and rearranging the terms gives

$$(1 - \rho_0)\left(V_{g/1} - V_{m/1}\right) \geq \rho_0\,(S - 1)\,\kappa\left(V_{m/1} - p + \psi\right) \geq 0.$$

Nevertheless, by equations (2) and (5), $V_{m/1} > V_{g/1}$ since $S > 1$ and $V_2 > 0$. This is contradiction. Q.E.D.

Proof of Proposition 3

Using $\max\left\{V_{m/1} - p, V_{m/0}\right\} = V_{m/1} - p$ and solving equations (3) and (5) for V_2, $V_{m/1}$ and $V_{g/1}$ give

$$V_2 = \frac{Y - \rho_2(p + C_1)}{1 - \rho_1\rho_2} \text{ and} \tag{43}$$

$$V_{m/1} = \rho_1\frac{Y - \rho_2(p + C_1)}{1 - \rho_1\rho_2} - C_1. \tag{44}$$

Sticking V_2 derived in equation (43) into equation (2) gives

$$V_{g/1} = \frac{\rho_1}{S} \frac{Y - \rho_2(p + C_1)}{1 - \rho_1 \rho_2} - C_1. \tag{45}$$

Sticking $V_{g/1}$ derived in equation (45) into equation (12) and solving for p give equation (19).

To make sure that the value of the firm in every stage is nonnegative, we just need to check if $V_{g/0}$ is nonnegative because it is the smallest of all. Sticking the equilibrium price p into $V_2, V_{m/1}$ and $V_{g/1}$ gives the firm values expressed only by exogenous variables as follows.

$$\begin{aligned} V_2^* &= \frac{S\,(Y - \rho_2 \psi)}{S\,(1 - \rho_1 \rho_2) + \rho_1 \rho_2}, \\ V_{m/1}^* &= \frac{\rho_1 S\,(Y - \rho_2 \psi)}{S\,(1 - \rho_1 \rho_2) + \rho_1 \rho_2} - C_1, \text{ and} \end{aligned}$$

$$V_{g/1}^* = \frac{\rho_1\,(Y - \rho_2 \psi)}{S\,(1 - \rho_1 \rho_2) + \rho_1 \rho_2} - C_1.$$

Further, sticking $V_{g/1}^*$ into equation (7) gives

$$V_{g/0}^* = \frac{\rho_0 \rho_1\,(Y - \rho_2 \psi)}{S\,(1 - \rho_1 \rho_2) + \rho_1 \rho_2} - \rho_0 C_1 - C_0.$$

This proves equation (14) as a necessary condition.

It remains to show that an established firm is willing to pay the equilibrium price to start developing a project immediately, instead of incubating a project by itself. To show this, we need to separately examine two cases depending on which $V_{m/1}^* - p$ or $p - \psi$ is bigger, since the equilibrium value of $V_{m/0}$ depends on this comparison.

First, since

$$\left(V_{m/1}^* - p\right) - (p - \psi) = \frac{(S - 2)\,\rho_1 Y - (S - \rho_1 \rho_2)\,\psi}{S\,(1 - \rho_1 \rho_2) + \rho_1 \rho_2} + C_1,$$

if equation (15) is satisfied, evaluating $V_{m/0}$ gives

$$V_{m/0} = \rho_0 V_{m/1} - (S - 1)\,\kappa \rho_0 p - C_0.$$

Then,

$$
\begin{aligned}
& V_{m/1} - p - V_{m/0} \\
= \quad & \frac{(1-\rho_0) S - (1-(S-1)\kappa\rho_0)}{S(1-\rho_1\rho_2)+\rho_1\rho_2}\rho_1 Y - \frac{1-\rho_0(\rho_1\rho_2+(S-1)\kappa(1-\rho_1\rho_2))}{S(1-\rho_1\rho_2)+\rho_1\rho_2} S\psi \\
& +\rho_0(1-(S-1)\kappa) C_1 + C_0
\end{aligned}
$$

Thus, equation (16) is a necessary condition.

Second, if

$$\left(V^*_{m/1} - p\right) - (p-\psi) < 0,$$

that is, equation (17) holds, then evaluating $V_{m/0}$ gives

$$V_{m/0} = (1-(S-1)\kappa)\rho_0 V_{m/1} + (S-1)\kappa\rho_0(p-\psi) - C_0.$$

Therefore,

$$
\begin{aligned}
& V_{m/1} - p - V_{m/0} \\
= \quad & \frac{\left(S-1-\left(S-(S-1)^2\kappa\right)\rho_0\right)\rho_1 Y - \left(S-\left(S-(S-1)^2\kappa\right)\rho_0\rho_1\rho_2\right)\psi}{S(1-\rho_1\rho_2)+\rho_1\rho_2} \\
& +\rho_0 C_1 + C_0.
\end{aligned}
$$

This establishes equation (18) as a necessary condition. Q.E.D.

Proof of Proposition 4

Solving the system of equations (20)-(25) gives the stationary distribution of firms expressed by the set of equations (46).

$$
\begin{aligned}
G_0 &= \frac{1}{\mu+\lambda_0} N, \\
G_1 &= \frac{\lambda_0 S\mu(\mu+\lambda_1+\lambda_2)}{(\mu+\lambda_0)(\mu+\lambda_1)\{\lambda_1\lambda_2+S\mu(\mu+\lambda_1+\lambda_2)\}} N, \\
M_0 &= 0, \\
M_1 &= \frac{\lambda_0\lambda_1\lambda_2}{(\mu+\lambda_0)(\mu+\lambda_1)\{\lambda_1\lambda_2+S\mu(\mu+\lambda_1+\lambda_2)\}} N, \text{ and} \\
M_2 &= \frac{\lambda_0\lambda_1}{(\mu+\lambda_0)\{\lambda_1\lambda_2+S\mu(\mu+\lambda_1+\lambda_2)\}} N, \qquad (46)
\end{aligned}
$$

Therefore, the rate of acquisition is

$$H_g = \frac{\lambda_2 M_2}{\lambda_0 G_0} = \left(1 + \frac{\mu S \left(\mu + \lambda_1 + \lambda_2\right)}{\lambda_1 \lambda_2}\right)^{-1},$$

which is decreasing in μ and S, and increasing in λ_1 and λ_2. Q.E.D.

Proof of Proposition 5

It is obvious that both $V_{g/0}$ and $V_{g/1}$ are decreasing in S. Differentiating $V_{m/1}$ and V_2 by S gives

$$\frac{dV_{m/1}}{dS} = \frac{\rho_1^2 \rho_2 \left(Y - \rho_2 \psi\right)}{\left(S\left(1 - \rho_1\rho_2\right) + \rho_1\rho_2\right)^2} > 0$$

and

$$\frac{dV_2}{dS} = \frac{\rho_1 \rho_2 \left(Y - \rho_2 \psi\right)}{\left(S\left(1 - \rho_1\rho_2\right) + \rho_1\rho_2\right)^2} > 0.$$

Therefore, the proposition follows. Q.E.D.

Proof of Proposition 6

Note that equations (43), (44), and (45) all hold, since $V_{m/1} - p \geq V_{m/0}$. Sticking $V_{m/1}$ obtained in equation (44) into equation (4) and using $p - \psi = \max\left\{p - \psi, V_{m/0}, V_{m/1} - p\right\}$ give

$$V_{m/0} = \left(1 - \left(S - 1\right)\kappa\right)\rho_0 \frac{\rho_1 Y - C_1 - \rho_1\rho_2 p}{1 - \rho_1\rho_2} + \left(S - 1\right)\kappa\rho_0\left(p - \psi\right) - C_0.$$

In equilibrium, $V_{m/0} = V_{m/1} - p$. Solving this equation for p gives

$$p = \frac{\left(1 - \left(1 - \left(S - 1\right)\kappa\right)\rho_0\right)\left(\rho_1 Y - C_1\right)}{1 - \rho_0\rho_1\rho_2 + \left(S - 1\right)\kappa\rho_0} + \frac{\left(1 - \rho_1\rho_2\right)\left(C_0 + \left(S - 1\right)\kappa\rho_0\psi\right)}{1 - \rho_0\rho_1\rho_2 + \left(S - 1\right)\kappa\rho_0}.$$

To make sure that the value of the firm in every stage is nonnegative, we just need to check if $V_{g/0}$ is nonnegative because it is the smallest of all. Solving for $V_{g/0}$ gives

$$V_{g/0}^* = \frac{\rho_0\rho_1}{S}\frac{\left(1 + \left(S - 1\right)\kappa\rho_0\right)Y - \rho_2\left(\rho_0 C_1 + C_0 + \left(S - 1\right)\kappa\rho_0\psi\right)}{1 - \rho_0\rho_1\rho_2 + \left(S - 1\right)\kappa\rho_0} - \rho_0 C_1 - C_0.$$

This establishes equation (28) as a necessary condition.

It remains to show that equation (27) is satisfied. Valuating $V_m - p$, $p - \psi$, and $V_{g/1}$ at the equilibrium price p gives the necessary conditions (29) and (30). Q.E.D.

Proof of Proposition 7

Solving equations (32)-(36) and (37) for G_0, G_1, M_0, M_1, M_2 and H_m gives

$$G_0 = N/\left(\mu + \lambda_0\right),$$

$$G_1 = \lambda_0 G_0/\left(\mu + \lambda_1\right),$$

$$M_1 = \frac{\lambda_0\lambda_1\lambda_2}{\left(\mu + \lambda_1\right)\left(\mu + \lambda_2\right)\left(\left(\mu + \lambda_0\right) + \left(S - 1\right)\kappa\lambda_0\right) - \lambda_0\lambda_1\lambda_2}\frac{G_1}{S}$$

$$M_0 = \frac{\left(\mu + \lambda_1\right)}{\lambda_0}M_1$$

$$M_2 = \frac{\left(\mu + \lambda_1\right)\left(\left(\mu + \lambda_0\right) + \left(S - 1\right)\kappa\lambda_0\right)}{\lambda_0\lambda_2}M_1,$$

and

$$H_m = \frac{\left(S - 1\right)\kappa\lambda_0}{\left(\mu + \lambda_0\right) + \left(S - 1\right)\kappa\lambda_0}.$$

Since $\partial H_m/\partial\kappa > 0$, $\partial H_m/\partial\mu < 0$, and $\partial H_m/\partial\lambda_0 > 0$, the proposition follows. Q.E.D.

Proof of Proposition 8

To prove Proposition 8, we first prove the following two lemmas.

Lemma 1 *Suppose that the assumptions stated in Proposition 8 hold. Then, one of the four cases is true. The first case is*

$$p - \psi \geq \max\left\{V_{g/0}\left[N\right], V_{g/1}\left[N\right]\right\}$$

and therefore

$$V_{g/0}\left[N\right] = \rho_0\left(p - \psi\right) - C_0. \tag{47}$$

The second case is $V_{g/1}\left[N\right] \geq p - \psi$ *and* $V_{g/0}\left[N\right] \geq \max\left\{V_{g/1}\left[N\right] - p, p - \psi\right\}$, *and therefore*

$$V_{g/0}\left[N\right] = \frac{\left(1 - N\kappa\right)\rho_0 V_{g/1}\left[N\right] - C_0}{\left(1 - N\kappa\rho_0\right)}. \tag{48}$$

The third case is $V_{g/1}\left[N\right] \geq p - \psi$ *and* $V_{g/1}\left[N\right] - p \geq \max\left\{V_{g/0}\left[N\right], p - \psi\right\}$

$$V_{g/0}\left[N\right] = \rho_0 V_{g/1}\left[N\right] - N\kappa\rho_0 p - C_0 \tag{49}$$

The last case is $V_{g/1}[N] \geq p-\psi$ *and* $p-\psi \geq \max\left\{V_{g/0}[N], V_{g/1}[N]-p\right\}$, *and therefore*

$$V_{g/0}[N] = (1-N\kappa)\rho_0 V_{g/1}[N] + N\kappa\rho_0(p-\psi) - C_0. \tag{50}$$

Proof. There are two maximum brackets in the right hand side of equation (38). The first bracket has two entries and the second bracket has three entries. Therefore, in general, there are 2 times 3 cases depending on which entry is the maximum of each bracket. Nevertheless, note that if $V_{g/0}[N] \geq \max\left\{V_{g/1}[N]-p, p-\psi\right\}$, then $V_{g/1}[N] \geq p-\psi$.[5] Also note that if $V_{g/1}[N]-p \geq \max\left\{V_{g/0}[N], p-\psi\right\}$, then $V_{g/1}[N] \geq p-\psi$. Therefore, we can focus on the four cases stated in the lemma. ■

Lemma 2 *Suppose that the assumptions stated in Proposition 8 hold. Then,* $V_{g/i}[S-1] > V_{g/i}[S-2]$ *for* $i=0,1$.

Proof. A positive profit occurs only when a firm develops a project that matches with the firm's capability. The probability that this match occurs is strictly higher if an established firm is developing than otherwise. As a result, $V_{m/1} > V_{g/i}[N]$ for $N=0,...,S-2$ and $i=0,1$. We start with showing that $V_{g/i}(S-2) < V_{g/i}(S-1)$ for $i=1,2$. Let

$$\eta_0[N] = \max\left\{V_{g/0}[N], V_{g/1}[N]-p, 0\right\}$$

and

$$\eta_1[N] = \max\left\{V_{g/0}[N], V_{g/1}[N]-p, p-\psi\right\}.$$

Since $V_{m/1}-p \geq V_{m/0}$, $\eta_0[S-1] = \max\left\{V_{m/1}-p, 0\right\}$. According to the definition of $V_{m/1}$, $\rho_1 V_2 = V_{m/1} + C_1$. Thus, setting $N=S-2$ in equation (40) gives

$$\begin{aligned} V_{g/1}[S-2] &= \frac{1}{2}\rho_1 V_2 + \frac{1}{2}\rho_1\eta_0[S-1] - C_1 \\ &= \frac{1}{2}V_{m/1} + \frac{1}{2}\left(\rho_1 \max\left\{V_{m/1}-p, 0\right\} - C_1\right) \\ &< V_{m/1} = V_{g/1}[S-1]. \end{aligned} \tag{51}$$

[5] This is because if $V_{g/0}(N) \geq p-\psi$, then

$$(1-N\kappa)\rho_0(p-\psi) + N\kappa\rho_0 V_{g/0}(N) - C_0 < V_{g/0}(N).$$

Now we are going to prove that $V_{g/0}[S-1] > V_{g/0}[S-2]$. If $V_{g/0} = \rho_0(p-\psi) - C_0$, then $V_{g/0}[S-1] > V_{g/0}[S-2]$ obviously. Suppose not and $V_{g/0}[S-2] \leq \max\{V_{g/1}[S-2]-p, p-\psi\}$. Then, because $V_{g/1}[S-2] < V_{g/1}[S-1]$,

$$\eta_1[S-1] > \eta_1[S-2].$$

Given this fact, taking a difference of equation (38) and dividing it by ρ_0 give

$$\begin{aligned}
& \left(V_{g/0}[S-1] - V_{g/0}[S-2]\right)/\rho_0 \\
= \quad & \frac{(1-S\kappa)}{2}V_{m/1} - \frac{(1-(S-2)\kappa)\rho_1}{2}\eta_0[S-1] + \frac{(1-(S-2)\kappa)}{2}C_1 \\
& +(S-1)\kappa\eta_1[S-1] - (S-2)\kappa\eta_1[S-2]
\end{aligned}$$

$$\begin{aligned}
\geq \quad & \frac{(1-S\kappa)}{2}V_{m/1} - \frac{(1-(S-2)\kappa)\rho_1}{2}\eta_0[S-1] + \frac{(1-(S-2)\kappa)}{2}C_1 + \kappa\eta_1[S-1] \\
\geq \quad & \frac{(1-S\kappa)}{2}\left(V_{m/1} - \eta_0[S-1]\right) + \frac{(1-(S-2)\kappa)}{2}C_1 \\
> \quad & 0
\end{aligned}$$

The second last inequality holds because $\kappa\eta_1[S-1] \geq \rho_1\kappa\eta_0[S-1]$. According to Lemma 1, it only remains to show that $V_{g/0}[S-1] > V_{g/0}[S-2]$ when $V_{g/0}[S-2] \geq \max\{V_{g/1}[S-2]-p, p-\psi\}$. If the latter inequality holds then

$$V_{g/0}[S-2] = \frac{(1-(S-2)\kappa)\rho_0 V_{g/1}[S-2] - C_0}{1-(S-2)\kappa\rho_0}.$$

Since

$$\begin{aligned}
& V_{g/0}[S-2] \\
= \quad & (1-(S-1)\kappa)\rho_0 V_{m/1} + (S-1)\kappa\rho_0 \max\{V_{m/1}-p, p-\psi, V_{m/0}\} - C_0 \\
\geq \quad & (1-(S-1)\kappa)\rho_0 V_{m/1} + \kappa\rho_0\eta_1[S-1] + (S-2)\kappa\rho_0 V_{m/0} - C_0,
\end{aligned}$$

noting that $V_{g/0}[S-1] = V_{m/0}$, we have

$$V_{g/0}[S-1] \geq \frac{(1-(S-1)\kappa)\rho_0 V_{m/1} + \kappa\rho_0\eta_1[S-1] - C_0}{1-(S-2)\kappa\rho_0}.$$

Using equation (51), we have

$$\begin{aligned}
& \left(V_{g/0}[S-1]-V_{g/0}[S-2]\right)/\left(1-(S-2)\kappa\rho_0\right) \\
\geq\ & (1-(S-1)\kappa)\rho_0 V_{m/1}+\kappa\rho_0\eta_1[S-1]-(1-(S-2)\kappa)\rho_0 V_{g/1}[S-2] \\
=\ & (1-(S-1)\kappa)\rho_0 V_{m/1}+\kappa\rho_0\eta_1[S-1] \\
& -(1-(S-2)\kappa)\rho_0\left(\frac{1}{2}V_{m/1}+\frac{1}{2}(\rho_1\eta_0[S-1]-C_1)\right) \\
\geq\ & \frac{(1-S\kappa)\rho_0}{2}\left(V_{m/1}-\rho_1\eta_0[S-1]\right)+(1-(S-2)\kappa)\rho_0 C_1 \\
>\ & 0.
\end{aligned}$$

Thus, the proposition follows. ■

Lemma 3 *Suppose that the assumptions stated in Proposition 8 hold. Then,* $V_{g/1}[N]>V_{g/0}[N]$ *for* $N=0,1,...,S-1$.

Proof. By assumption $V_{g/1}[0]>V_{g/0}[0]$ and $V_{g/1}[N]>V_{g/0}[N]$. Therefore, it remains to prove the lemma for $N=1,...,S-2$. Examining equations (47)-(50) reveals that $V_{g/1}[N]\leq V_{g/0}[N]$ only if $V_{g/1}[N]\leq p-\psi$ and $V_{g/0}[N]=\rho_0(p-\psi)-C_0$. Suppose that N' is the highest N such that $V_{g/1}[N]\leq p-\psi$, that is, if $N>N'$, $V_{g/1}[N]>p-\psi$. The former condition implies that

$$V_{g/1}\left[N'\right]=\rho_1\left(\frac{1}{S-N'}V_2+\frac{S-N'-1}{S-N'}\eta_0\left[N'+1\right]\right)-C_1\leq p-\psi.$$

Then, since $V_{g/1}[N]<p$, $\eta_0[N']=\max\{\rho_0(p-\psi)-C_0,0\}$. As a result,

$$\eta_0\left[N'+1\right]>\eta_0\left[N'\right].$$

Therefore,

$$\begin{aligned}
V_{g/1}\left[N'-1\right] &= \rho_1\left(\frac{1}{S-N'+1}V_2+\frac{S-N'}{S-N'+1}\eta_0\left[N'\right]\right)-C_1 \\
&< \rho_1\left(\frac{1}{S-N'}V_2+\frac{S-N'-1}{S-N'}\eta_0\left[N'+1\right]\right)-C_1 \\
&= V_{g/1}\left[N'\right]\leq p-\psi.
\end{aligned}$$

Repeatedly applying this argument, we have $V_{g/1}[0] < p - \psi$, which is contradiction. ■

Now we are ready to prove Proposition 8 by induction. In particular, we are going to show that if $V_{g/i}[N] < V_{g/i}[N+1]$, then $V_{g/i}[N-1] < V_{g/i}[N]$. Solving equation (40) for $\rho_1 V_2$ gives

$$\rho_1 V_2 = (S-N)\left(V_{g/1}[N] + C_1\right) - (S-N-1)\,\rho_1 \eta\,[N+1]\,.$$

Sticking $\rho_1 V_2$ obtained this way into equation (40) gives

$$\begin{aligned}
& V_{g/1}[N] \\
= \; & \frac{1}{S-N}\rho_1 V_2 + \frac{S-N-1}{S-N}\rho_1 \eta_0[N+1] - C_1 \\
= \; & \frac{1}{S-N}\left((S-N-1)\left(V_{g/1}[N+1] + C_1\right) - (S-N-2)\,\rho_1 \eta_0[N+2]\right) \\
& + \frac{S-N-1}{S-N}\rho_1 \eta_0[N+1] - C_1 \\
= \; & \frac{S-N-1}{S-N}V_{g/1}[N+1] + \frac{1}{S-N}\rho_1 \eta_0[N+1] \\
& - \frac{S-N-2}{S-N}\rho_1\left(\eta_0[N+2] - \eta_0[N+1]\right) - \frac{1}{S-N}C_1. \qquad (52)
\end{aligned}$$

From Lemma 3, $V_{g/1}[N+1] > \eta_0[N+1]$. And if $\eta_0[N+2] > \eta_0[N+1]$, then $V_{g/1}[N] < V_{g/1}[N+1]$. Since $\eta_0[S-1] > \eta_0[S-2]$ from Lemma 2, this proves that $V_{g/1}[S-3] < V_{g/1}[S-2]$. Now it remains to prove that $V_{g/0}[N-1] < V_{g/1}[N]$ for $N = S-2$. Given that $V_{g/1}[N] > p - \psi$ due to Lemma 3, taking difference of equation (38) gives

$$\begin{aligned}
& \left(V_{g/0}[N] - V_{g/0}[N-1]\right)/\rho_0 \\
= \; & (1-N\kappa)\,V_{g/1}[N] + N\kappa\eta_1[N] - (1-(N-1)\,\kappa)\,V_{g/1}[N-1] \\
& - (N-1)\,\kappa\eta_1[N-1]\,,
\end{aligned}$$

sticking $V_{g/1}[N-1]$ from equation (52) and rearranging the terms give

$$\begin{aligned}
= \; & (1-N\kappa)\,V_{g/1}[N] - (1-(N-1)\,\kappa)\left(\frac{S-N}{S-N+1}V_{g/1}[N] + \frac{1}{S-N+1}\rho_1\eta_0[N]\right) \\
& + N\kappa\eta_1[N] - (N-1)\,\kappa\eta_1[N-1] \\
& + (1-(N-1)\,\kappa)\left(\frac{S-N-1}{S-N+1}\rho_1\left(\eta_0[N+1] - \eta_0[N]\right) + \frac{1}{S-N+1}C_1\right)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1-S\kappa}{S-N+1}V_{g/1}[N]-(1-(N-1)\kappa)\frac{1}{S-N+1}\rho_1\eta_0[N] \\
& +N\kappa\eta_1[N]-(N-1)\kappa\eta_1[N-1] \\
& +(1-(N-1)\kappa)\left(\frac{S-N-1}{S-N+1}\rho_1(\eta_0[N+1]-\eta_0[N])+\frac{1}{S-N+1}C_1\right).
\end{aligned}
$$

Therefore, if $\eta_0[N+1] \geq \eta_0[N]$, then since $\eta_1[N] \geq \rho_1\eta_0[N]$,

$$
\begin{aligned}
& \left(V_{g/0}[N]-V_{g/0}[N-1]\right)/\rho_0 \\
\geq \quad & \frac{1-S\kappa}{S-N+1}\left(V_{g/1}[N]-\rho_1\eta_0[N]\right)+(N-1)\kappa(\eta_1[N]-\eta_1[N-1]) \\
= \quad & \xi[N].
\end{aligned}
$$

Note that $V_{g/1}[N] > \rho_1\eta_0[N]$ because $V_{g/1}[N] \geq V_{g/0}[N]$ due to Lemma 3 and $p \geq 0$. Suppose $\eta_1[N-1] \neq V_{g/0}[N-1]$. Then, $\eta_1[N] \geq \eta_1[N-1]$ if $V_{g/1}[N] \geq V_{g/1}[N]$, therefore $V_{g/0}[N] > V_{g/0}[N-1]$. Suppose $\eta_1[N-1] = V_{g/0}[N-1]$. Then, since $\eta_1[N-1] \geq V_{g/0}[N-1]$,

$$
\xi[N] \geq (N-1)\kappa\left(V_{g/0}[N]-V_{g/0}[N-1]\right).
$$

Since $\left(V_{g/0}[N]-V_{g/0}[N-1]\right)/\rho_0 > \xi[N]$,

$$
\left(\frac{1}{\rho_0}-(N-1)\kappa\right)\left(V_{g/0}[N]-V_{g/0}[N-1]\right) > 0.
$$

Since $\left(\frac{1}{\rho_0}-(N-1)\kappa\right) > 0$,

$$
V_{g/0}[N] > V_{g/0}[N-1].
$$

Q.E.D.