"ECONOMICS OF CULTURE, INSTITUTIONS, AND CRIME"

Hosted by Fondazione Eni Enrico Mattei (FEEM)

Supported by
FP6 Priority 7 "Citizens and governance in a knowledge-based society" Project: Sustainable Development in a Diverse World"(SUS.DIV) (Contract No. CIT3-CT-2005-513438)

University of Padua Research Project "Economic analysis of crime and social interactions" (grant CPDA071899)

Fondazione Eni Enrico Mattei (FEEM)

Organized by
SUS.DIV, FEEM, University of Padua and CEPR

Milan; 20-22 January 2010

'Driving While Black': A Theory for Interethnic Integration and Evolution of Prejudice

Fabrizio Panebianco

We are grateful to the following institutions for their financial and organizational support: SUS.DIV, FEEM, University of Padua and CEPR.

'Driving While Black': A Theory for Interethnic Integration and Evolution of Prejudice*

Fabrizio Panebianco
Advanced School of Economics
Ca' Foscari University - Venice†

Abstract

This paper studies the evolution of interethnic attitudes, the integration or segregation dynamics of ethnic minorities and the conditions for the rising of ethnic-based social hierarchies. By means of a cultural evolution framework, a dynamics of interethnic attitudes is provided and conditions for their convergence derived. Steady states implying a constant role of racism and no role for racism are identified. By deriving sufficient conditions for convergence, we find that the way in which *Oblique Socialization Schemes* (the way children react to out-of-family stimuli when forming their cultural values) are defined and modelled becomes crucial for the structure of the derived long run equilibria. In particular, we find that Steady States implying an Ethnic-based social ranking or full integration of ethnicities may be reached depending on whether or not agents use Reciprocity and/or Ethnocentrism in their interethnic attitudes formation schemes. We study the conditions under which one group puts more effort in the socialization process, it changes more in values and shows more frustration than others. At last, we provide an endogeneization of socialization process by applying an homophily rule, finding out when breaks in the convergence process happen.

Keywords: Cultural transmission, Minority integration, Evolution of Preferences

JEL Classification: D10, J15, Z1

*I am really grateful to Alberto Bisin, Sergio Currarini, Piero Gottardi, Michael Pichler and Pier Luigi Sacco for their comments. The usual disclaimer applies. A preliminary version of this paper has been circulated under the same title as working paper in University Ca' Foscari of Venice, Dept. of Economics Research Paper Series No. 10/WP/2009

†e-mail: fabrizio.panebianco@unive.it

1 Introduction

Interactions among different ethnicities in modern societies have always been a great concern for many academics and politicians. The United States has been the first country to experience problems with interracial relationships, since the American society has always been composed by people of different ethnicities. Now Europe is also starting to encounter problems and opportunities deriving from a multicultural and multiethnic society. Moreover, given the actual rates of immigration, we can reasonably think that these issues will become increasingly important for the Western societies.

Thus, this paper studies the evolution of interethnic attitudes, the integration or segregation dynamics of ethnic minorities and the conditions for the rising of ethnic-based social hierarchies by means of a cultural evolution framework. Events as the banlieue riots in Paris in 2005 or the more recent election of Barack Obama as president of the United States, reopened a strong debate about integration of minorities: how much they get integrated and accepted in the society or how much they resist to integration in order to preserve their identity. Moreover, anti-immigrant political parties in some European countries rised the problem of native people worrying for their weaker position in the society with respect to some decades ago. As economist, it is important to answer to these questions since they are at the basis of some works on, for example, marriage markets, migration studies and spatial segregation. With respect to the first case we will describe later how Bisin et al. (2004, 2006) proposed models for intergroup marriages in which the intergroup preferences are one of the key elements of the analysis. In this respect, taking care of how interethnic attitudes change along time may be an important element in explaining the dynamics of interethnic marriages. A theory of the evolution of interethnic attitudes may also be internalized in immigration models in order to analyse if this element may be an important factor contributing to analysing immigration flows: why people decide to enter in a country or why they decide to leave. Interethnic attitudes have also been shown to be crucial in job hiring processes, as found by Bertrand and Mullainathan (2004), Carlsson and Rooth (2007) and Rooth (2009). In spatial segregation theories, as, for example, the basic Schelling (1971, 1978) model of spatial segregation, the preference or tolerance towards other groups is a crucial element: having a theory that analyse how these elements changes endogenously with the composition of the neighbourhood can bring to interesting results.

Other policy issues are related with ethnic and racial discrimination: for example the debate about the *Affirmative Actions* that favours some minorities in the access to jobs and schools rises some legal problems and debates among law scholars (for a review, Casadei and Re (2007)) with the emergence of new directions in legal studies as with the *Critical Race Theory* (Thomas and Zanetti (2005)), thus stressing the cultural origin of race and ethnic categorizations. Again, security practices as the *Racial Profiling Techiques* in order to discriminate potentially dangerous agents, selecting people to be controlled on the basis of race, rises lots of concernes, especially now that these techniques are used after the 9/11 (for a review, Goldoni (2007)). Even before 9/11 this was a procedure used in some security controls: for example black people complained that police agents stopped many black male drivers without any evident justification but race, ironically saying that the police stopped the person because he was *'driving while black'*. Consequently, as it is clear, understanding better the mechanisms that govern these interethnic attitudes dynamics may thus be crucial in the promotion of policies for integration.

Some sociological studies find the existence of ethnic hierarchies in the society, meaning that the society converges to an agreement over attitutes towards the ethnic groups. From an empirical point of view, this intergroup consensus over ethnic hierarchies has been studied for US (Bobo and Zubrinsky (1996); Duckitt (1992)), Canada (Berry and Kalin (1979, 1996)), Sweden (Snellman and Ekehammar (2005)), the Netherlands (Verkuyten and Kinket (2000)), as pointed out by Listhaug and Strabac (2008) that provide the same evidence for Muslim minorities. From a theoretical point of view Hagendoorn et al. (1995) explain why we observe ethnic hierarchies: this literature identifies the causes of this evolution of hierarchy in the process of prejudice formation, in a form of cultural distance among groups and in the socio-economic status of the group. This kind of studies try to understand why, in a society, a rank of the different ethnic groups could be observed so that a sort of agreement on most preferrable ethncities arises. These works rise important questions about the long run role of racism: is the ethnic social ranking we observe stable enough for ethnicity to always play a role in people' choices? Or under which conditions ethnic groups may agree on common attitudes towards anyother such that an 'end of racism' may be observed?

Even though some theoretical economic literature uses network theory in order to understand segregation and its

determinants (Jackson (2006) and Currarini et al. (2009)), here we focus on a second line of research: cultural evolution. Cultural evolution theories have their roots in the seminal work of Cavalli-Sforza and Feldman (1973, 1981) and Boyd and Richerson (1985, 2005). These theories develop theoretical models trying to capture the dynamics involved in the evolution of a given cultural trait. They are based on the intergenerational transmission and modification of some characters having as peculiar element the coevolution of biological and cultural traits. These models, originally developed by anthropologists and genetists, had recently been applied by economists in order to explain marriage choices in diverse societies. We use these theories since they provide instruments to analyse how values are formed, and how they may spread in the society, taking care of the interaction between this process and the environment in which the agents live: we think that these are key elements in the study of a social phenomenon as the evolution of interethnic attitudes. Moreover, given the fact that attitudes may be transmitted from one person to the other through a kind of imitation, and, thus, they are susceptible of modification, they can be considered as cultural traits without problems. Therefore in the rest of the paper we sometimes refer to the attitude as 'cultural trait' and to the set of the attitudes of a given ethnic group as 'culture' since the framework we proposed can be extended to the study of other cultural traits than interethnic attitudes.
The first works trying to introduce these concepts in the economic debate has been Bisin and Verdier (2000, 2001) in which the transmission of a cultural trait is modelled, and the dynamics of groups population is analyzed. In particular agents in these models are allowed to socialize their children to their cultural trait. These works represents the starting point to understand different phenomena about interethnic and religious intermarriage using random search models. The most interesting contributions are Bisin et al. (2004, 2006) in which models for religious intermarriages in the US and interethnic preferences in UK are set up. In particular, in Bisin et al. (2004) this model was used in order to estimate the intensity of ethnic identities depending on the social context children are rised.
In these last contributions, however, the cultural traits that are transmitted from one generation to the other, are fixed, so that only an analysis of the demographic trends is possible. Brueckner and Smirnov (2007, 2008) start to introduce the possibility of a change in the intensity of the cultural traits providing some sufficient conditions for convergence to a *Melting Pot* equilibrium. An innovating contribution is given by Pichler (2009) in which, in a reinterpretation of the Bisin and Verdier framework, parents can choose which kind of cultural trait transmit to their children and in which cultural values also evolve in intensity during time too, thus introducing richness in the modelization of the vertical socialization.

This paper goes a little bit further in the analysis, focusing on the role of Oblique Socialization. We define as *'integrated'* two ethnic groups that share the same attitudes towards any ethnic group, *'segregated'* when this does not happen, and differences in attitudes are observed, while *integration* is defined as the process that brings to integrated groups: in our case integration does not mean that two groups have good attitude towards each other, but just that their attitude vector is equal so that their cultural traits are identical. Moreover, we consider fixed ethnicities, so that we are not interested in how and if a melting pot society or mixed identities arise, but only under which conditions different cultural groups converge in attitudes still remaining distinguished. We use, as starting point, the Bisin-Verdier framework in which agents choose how much to socialize children. However, differently from these previous studies, we consider two cultural traits that are contemporarily involved in the dynamics: *'ethnicity'* and *'attitudes'* towards other ethnic groups. Since cultural evolution regards interaction between biological and cultural traits, in our study ethnicity is biologically determined and thus fixed but transmittable, while attitudes are culturally derived and thus are transmitted and changed in the socialization process, so that they are no more fixed[1]. Given this framewok we then consider what is said by Boyd and Richerson (1985) in the first pages of their first contribution to cultural evolution theories in which they argue that a theory for cultural evolution '*should predict the effect of different structures of cultural transmission on the evolutionary process*'. In particular, starting from different schemes of cultural transmission, we derive conditions under which ethnic social rankings, as previously defined, arise in the long run, and when attitudes converge to the same value, providing theoretical answers to these sociological questions, and also understanding when racism may be endogenous in these cultural dynamics. Consequently, we explain ethnic social hierarchies by using the first factor Hagerdoorn et al. (1998) uses: prejudice formation. In order to understand these

[1] In this sense, both Pichler (2009) and this work introduce, in different framweorks, the changing in the intensity of cultural traits as a problematic issue in cultural evolution models in economics.

dynamics, we study deeper a key element of cultural evolution theories: the socialization mechanism. In particular, depending on how children react to out of family stimuli (Oblique Socialization), the integration/segregation result may change. We then analyse when a group changes in values faster than others, when it pays much more attention to socialization and when its memebers are much more frustrated than the other groups' member, finding some condition for the policymaker in order to have faster integration and lower frustration in each group. Moreover we provide an analysis of what happens if groups differ in the use of oblique socialization schemes and derive conditions over the interethnic relational structures in order to get again ethnic hierarchies or a deeper integration. In all these cases, we underline the roles that a group may play in the intergroup relationships (cultural bridges or cultural hub) and the role of particular socialization schemes (reciprocity and ethnocentrism) understaning their impact on the long run outcome. As a last point, we provide a first insight for time-dependent socialization schemes, focusing on some conditions for convergence to long-run equilibria, thus opening a road towards the endogeneization of socialization mechanisms, as proposed in section 7. In this last case we analyse agents using cultural distance in order to form their network, and we study when this homophily rule brings to integration and when it does not.

Given the nature of this analysis, strong relationship with the theory on the spread of opinion in a network (DeGroot (1974), De Marzo et al. (2003) and Golub and Jackson (2007)) can be found. From a formal point of view we will point out, time by time, the differences in the mathematical structure between this work and De Marzo et al. (2003) which is the closest one from a mathematical point of view and, for section 7, with Golub and Jackson (2009). This relationship between the two theories makes clear that the interethnic attitude problem may be only one aspect of the analysis, and that this framework may be frutifully extended in the direction of a theory of opinion formation in a network. The rest of the paper has the following structure: in section 2 we describe the model, in section 3 we provide a general dynamics for interethnic attitudes as cultural traits studying conditions for convergence. Section 4 introduces the oblique socialization structures, while section 5 studies what happens if different groups uses different socialization structures. Section 6 introduces time-dependent oblique socialization structures. Section 7 ends the paper. Appendix A provides clarification over some numerical simulations, while Appendix B is devoted to the proofs of the propositions.

2 The Model

Consider a population composed of infinitely many agents. Every agent is characterized by the ethnicity, so that we can identify n ethnic subpopulations on the set $V \equiv \{i, j, k, ...w\}$. All the agents of a given ethnicity are supposed to be equal.

Agents of a given ethnicity i are also characterized by a vector $V_t^i \in [0,1]^n$, that we call '*type*', such that every entry is a coefficient associated to an ethnic group. Below an example for a 4 ethnicities world.

$$V^i := \begin{bmatrix} V^{ii} \\ V^{ij} \\ V^{ik} \\ V^{iw} \end{bmatrix} \quad \mathrm{V}^j := \begin{bmatrix} V^{ji} \\ V^{jj} \\ V^{jk} \\ V^{jw} \end{bmatrix} \quad \mathrm{V}^k := \begin{bmatrix} V^{ki} \\ V^{kj} \\ V^{kk} \\ V^{kw} \end{bmatrix} \quad \mathrm{V}^w := \begin{bmatrix} V^{wi} \\ V^{wj} \\ V^{wk} \\ V^{ww} \end{bmatrix}$$

All agents belonging to the same ethnic group have the same type. This vector is supposed to be observable and common knowledge. The element on the j^{th} position, for example V_t^{ij}, represents the attitude i agents have towards j agents. This may be seen as an objective index that measures the attitude each ethnic group has towards any other. Normalization on the [0,1] interval is arbitrary and using different normalizations do not change the result. The idea is that 0 correspond to the worst attitude possible and 1 to the best attitude possible. Examples of this measures are the ones by Golebiowska (2007) in which measures of reciprocal tolerance are derived by opinion surveys focusing on interpersonal trust and other social indicators. Other examples are derived by ethnic hierarchies studies in the social psychology field, as Hagendoorn et al.(1998), Listhaug and Strabac (2008), Berry (2006) and Schalksoekar et al. (2004), in which indexes that indicate the attitudes among groups are estimated so that an overview on how ethnic hierarchies may arise is given. This will be important in the next sections since we will consider the insurgence of ethnic hierarchies as a possible long run equilibrium of the society. Such studies become now much more easier by means of surveys as the World Value Survey or the International Study of Attitudes Towards Immigration and Settlement (ISATIS), that make the objectivation of these measures possible. A theoretical paper studying how these prejudices exist and are transmitted is given by Bar-Tal (1997), in which the roles of context, socialization and individual variables are examined[2]. In this way the measure of how much each group is tolerant towards any other, or has prejudice towards the others, is rendered objective and can be thought as known by every member that constantly lives in the society and interact daily with all the other members. Thus, this measure is considered as common knowledge since is derived from all the interactions that happen in the society.

Given these priors, the structure of the model is the following: every agent, at time t, reproduces asexually[3], so that a child is born from every agent. The child has the same ethnicity of the parent, but has no 'type' formed yet[4]. Thus, the parent produces a socialization effort τ_t^i in order to influence the child type. In particular, the parents would like to perfectly transmit their type to children, otherwise they experience a loss. However, socialization is costly. The child type then is formed considering the effect of the parental (or Vertical) socialization, and the societal (or Oblique) socialization. Oblique Socialization, in particular, is how other adults with well formed types influence the children' socialization process. After this socialization process has taken place, children become adults with defined types, can reproduce and start again the socialization of their children. Thus a dynamics of cultural traits is endogenously derived.

Parents try to transmit to their children their own type by producing a socialization effort: call $\tau_t^i \in [0,1]$ the effort parents i produce at time t in the Vertical Socialization. Children type will thus be given by Vertical and Oblique Socialization forces following the standard rules of socialization, as derived from cultural evolution literature:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1-\tau_t^i)\bar{V}^{ij} \tag{1}$$

in which $\tau_t^i V_t^{ij}$ is the vertical socialization part of the process and $(1-\tau_t^i)\bar{V}^{ij}$ is the oblique socialization part, with $\bar{V}^{ij} \in [0,1]$ being related to how society affects the j^{th} element of i agents belonging to $t+1$ generation. We suppose $\bar{V}^{ij}$ to be independent from τ_t^i. We call this a *Cavalli-Sforza Feldman Socialization Dynamics* (from now on CSF Dynamics)[5]. For the time being we do not characterize the Oblique Socialization.

Call W_t^i the utility agent i derives from having a child. Recalling that parents are happier the more effectively they can trasmit their type to their children, we can thus have the following:

$$W_t^i = V^* - \sum_k (V_t^{ik} - V_{t+1}^{ik})^2 \tag{2}$$

It simply states that if the child has the same values in all the type vector entries as the parent, then the parent has the highest possible utility from the child, V^*. Otherwise, he additionally experiences a loss dependent on the difference of the values. The parent also experiences a cost $c(\tau_t^i)$ for the effort produced. Consequently each parent will face the following

$$Max_{\tau_t^i}\ W_{t,i}^i - c(\tau_t^i)$$

[2]Another set of studies that uses these indexes are derived from the Bogardus Social Distance Scale (Bogardus (1926, 1959)). These studies, mainly referred to social psychology and psychometric techniques and developed in Hraba et al. (1999), Randall and Delbridge (2005), Lee et al. (1996) and Parillo and Donoghue(2005), estimate, by means of scaling systems, social dystance measures and indicators of how much groups reciprocally like.

[3]The model can be extended to the case of sexual reproduction following the matching as in Bisin and Verdier (2004, 2006). In this paper the model is kept as simple as possible in order to analyse only the effect of oblique socialization structures.

[4]Even though a pure genetic derivation of ethncity may be questionable, we use this simplificative approach in order to observe what happens to groups that do not experince mixed identities. Moreover an analysis of data as IPUMS (Integrated Public Use Microdata Series, for the US) show that this genetic approximation follows the data: self-assessments of ethnicity generally follows the ethnicity belonging of the parents, when both parents declare to belong to the same ethnic group.

[5]This equation has been used, in different forms, by Bisin and Verdier (2003) when introducing the possibility for the socialization of a continuous cultural trait. A more extensive use of this has been done in Pichler (2009), called, in that framework, *parental socialization techniques*. The first insights of this formulation can be found in Cavalli Sforza and Feldman (1981) when analysing the cultural transmission for a continuous trait, in chapter 5.

Define $\Delta\bar{V}_t^i \equiv \sum_k (V_t^{ik} - \bar{V}_t^{ik})^2$, thus representing the difference between the effect of oblique socialization over all type entries and parents type and being a general measure of the parent's loss. Substitute the (1) into (2) and we get that the parent wants to maximize

$$V^* - (1-\tau_t^i)^2 \Delta\bar{V}_t^i - c(\tau_t^i) \tag{3}$$

Thus the role of τ_t^i is here more evident: the higher the effort, the lower the general loss of the parent, but the higher the cost associated with this effort. Moreover, unless $\Delta\bar{V}_t^i = 0$, the marginal utility of τ_t^i, which is equal to $2(1-\tau_t^i)\Delta\bar{V}_t^i$, is positive and decreasing at a constant rate and is zero at maximum socialization effort. Then,

Assumption 1: Assume that the socialization cost function has the following properties: $c(\tau) : [0,1] \mapsto \Re^+$, $c(\tau) \in \mathcal{C}^3$, $c'(\tau_t^i)|_{\tau_t^i=0} = 0$ and $c''(\tau_t^i) \geq 0$

We can now state the following:

Proposition 1. *If Assumption 1 holds, then* $\tau_t^{i*} = Argmax[W_{t,i}^i(\bar{V}_t, \tau_t^i, \bar{p}_t) - c(\tau_t^i)]$ *exists and is unique* $\forall t, i$. *Moreover if* $V_t^{ij} \neq \bar{V}^{ij}$ *for at least one* j*, then* $\tau_t^{i*} \in (0,1)$.

Proof. See Appendix B. □

Assumption 1 states that costs should be flat at zero socialization and have non-negative slope elsewhere. This not very demanding assumption ensures the formation of an internal optimal socialization effort. This result is supported from the evidence that both society and parents actually enter in the children socialization process and influence his values. Only if $V_t^{ij} = \bar{V}_t^{ij}$ then the family and the society have the same effect on the children' type so that, being the socialization costly, parents choose not to socialize children, since what children can take from the society is the same they can transmit to the offpring so that no incentives for vertical socialization are present.
The society we describe here is very conservative in the sense that no agent has utility derived from diversity, but everyone would like to have children with his own very same preferences. An usual explanation for this is that parents judge their offpring by means of their preferences so that they use what is called *'imperfect empathy'* (Bisin and Verdier (2000)); we will maintain this behavioural assumption along all this work[6].

3 Cultural Dynamics

In equation (1) the dynamics of the cultural traits crucially depend on how oblique socialization is defined since, depending on it, parents experiences different losses and thus may choose different socialization efforts. In particular $\bar{V}^{ij}$ identifies the generic oblique socialization effect on the element V_{t+1}^{ij}. We analyse here this element.
The simplest way in order to intend oblique transmission of a cultural trait is taking the social average for that trait. This imply that the child randomly meets agents belonging to the parents' generation in the society and thus takes the average value from these encounters (for example teachers or other cultural models in the society). Equation (1) will thus become:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1-\tau_t^i)(\sum_k p_t^k V_t^{kj}) \tag{4}$$

[6]It has to be noted that these standard assumptions over socialization schemes imply that, since parents know the exact outcome of Oblique Socialization, then they can fully determine their children type and they are sure that their actions maximize their ex-ante and ex-post utilities. Moreover, in this simplified framework, children have only a passive role. Even though simplificative, for the time being we take these assumptions as true, just recalling the limits of this view since, in reality, children actually play an active role in their socialization process and there is also an element of uncertainty in oblique socialization that parents cannot control for, so that oblique socialization is subjected to a form of ambiguity or, at least, of randomness.

This can represent the most frictionless society we can imagine. For example the case in which children live in a neighborhood with no biases in group shares, or attend schools with professor of different ethnic groups in quota proportional to the population shares in the overall society or ethnic messages are reported by media respecting the proportions of ethnicities in the society. Moreover, unless $\tau_t^i = 1$, so that parents produce an enormous socialization effort (suboptimal under Assumption 1), it is impossible to have any V_t^{ij} fixed. This structure, then, restricts the possible influences of the Oblique Socialization since it is not possible that in the V_{t+1}^{ij} formulation process, i agents take care of the V_t^{kw} value, and it restricts the possible weights assigned to the different traits, fixing them equal to the population shares vector. We thus propose a more general formulation for the socialization dynamic:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{t,ij}^{kw} V_t^{kw}) \tag{5}$$

in which $w_{t,ij}^{kw}$ is a parameter, that for the time being we consider exogenous, simply stating if the agent consider the V_t^{kw} in the V^{ij} dynamic. This could be a measure of similarity of situations, of trust or other factors that could also be proportional to the population size of k. We consider these weights such that $0 \leq w_{t,ij}^{kw} \leq 1 \ \forall i, j, k, w$ and $\sum_{k,w} w_{t,ij}^{kw} = 1$. Thus the matrix of these parameters is row normalized and gives a full characterization of the oblique socialization technology at any time t.

In order to keep the model as simple as possible, for the time being we consider weights that are not time dependent so that the equation (5) becomes:

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{ij}^{kw} V_t^{kw}) \tag{6}$$

We call this cultural dynamics a *Generalized Cavalli-Sforza and Feldman Socialization Dynamics* (from now on GCSF Dynamics). This may be a first approximation of reality if population shares do not change during time or if the strucutre of the society (schools, neighborhoods, for example) are almost stable in time.

In order to choose an optimal socialization effort, the parents should know the weights vector their children are going to use in the oblique socialization effort and thus get a precise computation of all the social influences they get.

This formalization is similar to the one in De Marzo et al. (2003) in which they have each group having a single cultural trait influenced by the neighborhood. Their rule can be written as:

$$\bar{x}_{t+1} = T_t \bar{x}_t$$
$$T_t = (1 - \lambda_t)I + \lambda_t T.$$

in which $\bar{x}_t$ is the vector of values, T_t is the recursive rule, λ_t is a friction parameter that can recall our τ_t^i, and T is the time independent matrix indicating the network influences. From a mathematical point of view, our model extends this analysis with four progressive steps: we have that each group have more that one value involved in the dynamics (namely n^2 attitudes) instead of one, and thus a multidimensional dynamics arises. Then we allow the friction parameter to be not only time dependent but also group dependent. This will create a different bias for each group: in this framework this will be interesting since this correspond to the socialization effort whose implication are analysed in section 4.4. Finally, in section 6, we allow the matrix T to be time independent and, in section 7, to be endogenous with the model.

Given this we can state the following:

Proposition 2. *If Assumption 1 holds, then any* GCSF Dynamics *converges to a steady state.*

Proof. See Appendix B. ☐

This proposition basically states that if socialization is such that parents always have incentive to socialize their children at least a little bit, then convergence towards a steady state happens. Thus, the role of vertical socialization is to ensure convergence since, if for any reason $\tau = 0$ out of equilibrium, convergence may not happens and cycles may arise. Proposition 2 does not say which kind of steady state is reached and thus leaves the door open to different equilibria implying different levels of integration or segregation: this will be the topic of the next sections.
From a technical point of view proposition 2 also generalizes the contributions of Brueckner and Smirnov (2007, 2008) since here we control for parents' socializing role and for a wider range of possible interaction among ethnic groups, not restricting to the cases in which the matrix of relevance parameters forms irreducible matrices or block diagonal irreducible matrices, thus providing a more general sufficient condition for convergence. When, in section 6, we analyse time dependent weights, the generalization of previous theorems is more complete.
At steady state it happens that $\tau_t^{i*} = 0$. By the definition of steady state $V_t^{ij} = V_{t+1}^{ij}, \forall i, j, t$ so that parents and sons always have the same type. Consequently there is no incentive to socialize children since the loss the parents experience is zero. This implies that, in the long run, parents would not have any role in the socialization: since they care only about having children similar to them, once that this is an outcome of oblique socialization, they do not care anymore about it.

Remark: In the proof of proposition 2 we also show that even for the case of suboptimal socialization efforts, if $\tau_t^i, \forall i$ is strictly positive, convergence happens, even though steady state values may be different from the case in which the optimal τ_t^{i*} is chosen. Now, as we have argued above, in order to choose an optimal socialization effort the parent should know the whole matrix of all attitudes V, and the vector of weights $\bar{w}$ his son is going to use in the oblique socialization process. While the first assumption may be reasonable since the matrix of V is common knowledged, the second one may be questioned, since oblique socialization influences may not be perfectly predicted by parents. Still, even if the parent has a wrong guess of the relevance parameters, and thus choose an ex-post suboptimal socialization effort, if the chosen $\tau \in (0, 1]$, then convergence happens[7]. Then, far from being useless, different level of vertical socialization have effect on the levels of the steady state. Morevoer, the introduction of the optimal socialization effort makes the model richer, such that it will be useful in policy and welfare analysis that will be run in section 4.4.

4 Oblique Socialization Schemes and Evolution

4.1 Socialization Schemes

In the last section we proved that convergence to a steady state happens under some weak conditions. However, a crucial element of cultural dynamics is how children are influenced during their Oblique Socialization. As Boyd and Richerson (1985) explain in the first pages of their contribution to cultural evolution studies, '*the theory should predict the effect of different structures of cultural transmission on the evolutionary process*'. In our case a structure of cultural transmission is fully characterized by the structure of the oblique socialization weights matrix. Any vector of these weights identifies an *Oblique Socialization Scheme*. In order to follow the Boyd-Richerson approach, starting from simple socialization schemes we now derive long run equilibria that can be considered sensitive in the study of integration and segregation of groups, so that an analysis of how different socialization structures influence the process is the key element of the rest of the paper.
We find reasonable that, while forming their attitudes towards other groups, agents may use two basic schemes: Reciprocity and Ethnocentrism. With the first one we mean that people tend to form bad (good) attitudes towards people that have a bad (good) attitude towards them. With the second one we mean the possibility that a group have a good attitude towards people of their own ethnic group and never question this attitude. Looking at studies in sociology and social psychology it can also be found (Berry (2006) and Berry and Kalin (1979, 1996), for example) that agents

[7] From a mathematical point of view, $\tau_t^i > 0, \forall i$ make the diagonal entries of the transmission matrix A strictly positive. Thus, the matrix A, or its diagonal blocks, if irreducible, are also acyclic. However, it is not necessary to have $\tau_t^i > 0$ in order to have acyclic matrix, since an acyclic matrix may also derive from some particular structures of oblique socialization. However, since we have not put constraints on the oblique socialization scheme, $\tau_t^i > 0$ ensures acyclic matrix. This will become clear with the next sections.

actually use *Reciprocity* and *Ethnocentrism* in their attitude formation schemes. In particular, the correlation between inter-group attitudes has been computed and has been found positive so that Reciprocity seems to be an actual way of attitude formation; on the other side ethnocentrism has been proved to exists in all cases even though with different intensities depending on the ethnic group.
In terms of our model, Reciprocity means that $w_{ij}^{ji} > 0$, so that V_t^{ji} enters in the formation of V_{t+1}^{ij}, and if j has a bad (or good) attitude towards i, then i children take this into account while forming their attitudes. With Ethnocentrism we mean that agents do not question the reflexive attitudes, so that $w_{ii}^{ii} = 1$ and thus $V_t^{ii} = V_{t+1}^{ii}$.
We thus build 4 general socialization schemes in which there can be no reciprocity and no ethnocentrism, or one of the two or both. With respect to the equation (5) the following schemes are restrictions of the most general case since we impose particular structures on the oblique socialization weights matrix.

Definition 1: Call

- *Emulation Rule* an oblique socialization rule in which
  $(w_{ij}^{kw} = 0, \forall w \neq j, w_{ij}^{kj} > 0, \forall i, j, k, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_k w_{ij}^{kj} V_t^{kj}) \forall i, j, t;$
- *Ethnocentrism Rule* an oblique socialization rule in which
  $(w_{ij}^{kw} = 0, \forall w \neq j, w_{ij}^{kj} > 0, \forall i, j, k, w_{ii}^{ii} = 1, \forall i, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k \neq j} w_{ij}^{kj} V_t^{kj}) \forall i, j, t;$
- *Reciprocity Rule* an oblique socialization rule in which
  $(w_{ij}^{kj} > 0, \forall i, k, j, w_{ij}^{ji} > 0, \forall i, j, \text{ else } w_{ij}^{kw} = 0, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_k w_{ij}^{kj} V_t^{kj} + w_{ij}^{ji} V_t^{ji}) \forall j \neq i, \forall t;$
- *Reciprocity and Ethnocentrism Rule* an oblique socialization rule in which
  $(w_{ij}^{kj} > 0, \forall i, k, j, w_{ij}^{ji} > 0, \forall i, j, w_{ii}^{ii} = 1, \forall i \text{ else } w_{ij}^{kw} = 0, \sum_{k,w} w_{ij}^{kw} = 1, \forall i, j)$
  $V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k \neq j} w_{ij}^{kj} V_t^{kj} + w_{ij}^{ji} V_t^{ji}) \forall j \neq i, \forall t.$

The first scheme has neither Reciprocity nor Ethnocentrism so that the attitude V_{t+1}^{ij} depend on all the attitudes of everyone towards j.In the second case we introduce ethnocentrism so that $V_t^{ii} = V_{t+1}^{ii}$. Since $\tau_t^i \in (0, 1)$ under Assumption 1, in order to have this we impose that $w_{ii}^{ii} = 1, \forall i$. In this case all the other $V_t^{ij}, \forall i \neq j$ follow the previous rule. In the third case the reciprocity introduce the possibility of having $w_{ij}^{ji} > 0$ so that V_{t+1}^{ij} depends on the attitudes towards j plus the attitude of j towards i. The fourth case just combines the previous two situations. It should be underlined that these four rule are all symmetric in the sense that all agents of every group use the same rule: everyone use reciprocity towards anyother or no one uses reciprocity. An extension to asymmetric socialization rules will be done in the next section.

4.2 Steady State Characterization

In this subsection we focus on steady states, identifying 4 classes of them that may be considered benchmark outcomes of cultural dynamics: we consider them in relation to their integration or segregation properties.
As previously argued, in some literature there has been found evidence of social hierarchies based on ethnicity: in particular agents seem to agree on a ranking of different ethnicities, so that common prejudices arise. In terms of our model, if a common hierarchy is shown, we have that $\lim_{t \to \infty} V_t^{ij} = \lim_{t \to \infty} V_t^{kj}$ $\forall i, k$. We call these kind of steady states *Hierarchy Equilibria (HE)*. This situation may be represented by the following matrix in which every row is a type

vector so that the ij entry is the V_t^{ij}.

HE	i	j	...	k
i	a	b	c	d
j	a	b	c	d
...	a	b	c	d
k	a	b	c	d

Suppose for example that $a > b > c > d$ then there is an intergroup consensus on the fact that i ethnic groups is the *best* ethnic groups since everyone has the best attitude towards it. On the reverse k agents has a bad attitude towards themselves too, and are also considered the worst group among all.
A second kind of steady state is the one that predicts the 'end of racism' where $\underset{t\to\infty}{lim} V_t^{ij} = \underset{t\to\infty}{lim} V_t^{kw}\ \forall i, j, k, w$. If a steady state like this is reached, then a process by which all agents will end up with the same attitude towards every ethnic group has taken place. We call these outcomes *Integration Equilibria (IE)*. This equilibrium can be seen as the objective of integrationist policies. In this case, however, it does not happen that all groups merge in one single culture, but only that they do not discriminate among any culture. Every group, in fact, can continue to have its own cultural norms, and the society may continue to be formed by different cultural ways of life (since these cultural traits are not involved in this process), what converges here is just the attitutes groups reciprocally have. Thus, given our framework, this cannot be defined as a Melting Pot equilibirum. The second matrix represents this case.

IE	i	j	...	k
i	a	a	a	a
j	a	a	a	a
...	a	a	a	a
k	a	a	a	a

We should underline that Hierarchy and Integration Equilibria are not good states or bad states a-priori. With integration, in fact, we simply mean that all attitudes converge to the same value, so that it does not mean that these attitude should be good. It may happen that an IE is reached with very low final values, meaning that everyone has a bad attitude towards anyother, so that a very bad society is shown. On the reverse it can be that, in the first case, a ranking s shown but all the values are high, and thus represent very good attitudes. Since in these cases attitudes do not discriminate among ethnicities, we refer to them as *'racism-free'* outcomes. Thus, by the terms 'hierarchy' and 'integration', it is not meant any phenomenon with a specific positive or negative moral significance
In some situations it may not be the case that the groups which is considered as the worst one, is also self-considered bad. In order to control for this problem we add two more cases to the matrices above, implying that a sort of ethnocentric scheme, as for example found in Berry (2006) and Berry and Kalin (1979, 1996), may be considered. We can thus define Hierarchy Equilibria with Ethnocentrism (HEE) and Integration Equilibria with Ethnocentrism (IEE)(Suppose $V_t^{ii} = E\ \forall t, i$):

HEE	i	j	...	k
i	E	b	c	d
j	a	E	c	d
...	a	b	E	d
k	a	b	c	E

IEE	i	j	...	k
i	E	a	a	a
j	a	E	a	a
...	a	a	E	a
k	a	a	a	E

If we suppose that $E = 1$, then the attitude every ethnic group member has towards own ethnic group members is maximum. In this case an agreement on the attitude values holds, but everyone considers himself at the top of the ranking. The same may happen in the second case in which convergence of all attitudes towards the same value may happen, but still a form of ethnocentrism holds. In this last case, represented by the last matrix, every agent can only discriminate with his attitudes between members of his own groups and others out of the group.

4.3 Effect of Oblique Socialization Schemes

Given the previous definition of particular socialization schemes and steady states, we can now analyse if there are relations among those elements. Thus, we state the following:

Proposition 3. *A sufficient condition in order to get:*

- *HE is that Emulation Rule holds;*
- *IE is that Reciprocity Rule holds;*
- *HEE is that Ethnocentrism Rule holds;*
- *IEE is that Reciprocity and Ethnocentrism Rule holds;*

Proof. See Appendix B. □

With the last proposition we obtain an interesting result since, starting from two rules (Reciprocity and Ethnocentrism) important in the attitude formation schemes and derived from social psychology literature, we are able to prove convergence to four categories of steady states significant for their social properties. In particular we obtain that both racism-free and non racism-free steady states may be obtained given some condition over the oblique socialization schemes so that we can also state that racism may be a result of factors internal in the children socialization process, indirectly implying that policies that modify these schemes may have important results in term of racism outcomes. In particular a change in socialization structures such as reciprocity, widely changes the final outcome dramatically. We thus give reason of the intuition of Boyd-Richerson about the importance of the analysis of the cultural transmission schemes for the analysis of the long run equilibrium of the society.[8]
Figure 1 provides examples for the dynamics for the case of 3 ethnic groups in order to better understand what goes on. We consider here 3 ethnic groups and $c(\tau_t^i) = \tau_t^{i^2}$ as a simplest cost function satisfying the requirements of assumption 1. We then set up weights proportional to the population shares, using for simplicity $p^i = p^j = p^k = \frac{1}{3}$, but since population shares do not change, this is just a way to give a rule for the socialization weights. The matrix of the initial attitudes is:

$\bar{V}_0$	i	j	k
i	1	0.1	0.2
j	0.03	1	0.5
k	0.7	0.9	1

Thus a socialization dynamics as in equation (4) holds. Weight matrices are reported in Appendix A for all the four cases. Moreover we set up $E = 1$ in order to get the idea that groups may consider themselves as the best ones in the case in which ethnocentrism holds. However since E refers to type entries that do not experience any dynamics, this value can be change accordingly to cases. In the graphs the lines represents the attitudes and the dots the derived optimal socialization efforts for each ethnic group.
The top-left simulation regards a HEE equilibrium: after a very short adjustment we have at the top all the $V_t^{ii}, \forall i$, while each horizontal line then represents the attitude towards a specific ethnic groups, so that an ethnic hierarchy rises. The graph below shows the same socialization rule, but without ethnocentrism, so that also reflexive attitudes converge to the common ranking values. The graphs on the right represent, on the top a IEE and on the bottom a

[8] Additionally, proposition 3 makes clear that, if oblique socialization rules are such that $w_{ij}^{ij} > 0, \forall i, j$ parents do not play any role in the determination of the class of the long run equilibrium and convergence may happen without their contribution. However different socialization efforts will have an influence on the final levels. We will see, in the next section, that their presence may become important if a more general set of socialization rules is considered.

simple IE. In the first one we see that all attitudes, but reflexive ones, converge to the same value, while in the second one reflexive attitudes too converge to the common value.

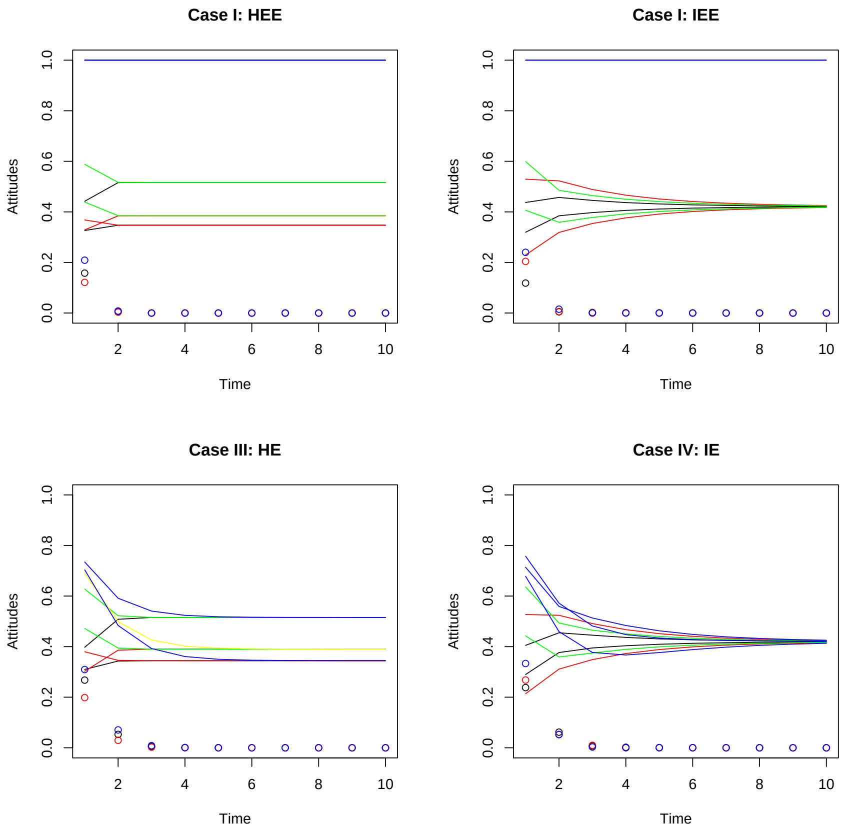

Figure 1: Simulations with fixed weights

It is interesting now to study if steady states classes are invariant under changes in the socialization schemes. In particular, assume that the steady state has been reached using the proper socialization scheme and that only the 4 socialization schemes previously defined may be used by agents.

Definition 2: We say that an IE (resp. IEE, HE, HEE) is invariant under a change in the socialization scheme if the new equilibrium under the new scheme is an IE (resp. IEE, HE, HEE).

We can thus state the following:

Corollary 1: *An IE is invariant under any change of socialization scheme, while a HEE is not invariant under any change of socialization scheme. An HE is invariant if ethnocentrism is added, while is not invariant if reciprocity is added. An IEE is invariant if reciprocity is removed and not invariant if ethnocentrism is removed.*

Corollary 1 states that in this framework, if only the 4 socialization schemes previously defined hold, once that an IE happened, no changes in the socialization process may alter the equilibrium class. On the contrary, if integration has not happened, then there is room for it to be reached if socialization schemes change appropriately. In particular, starting from a HEE, then any of HE, IEE, IE may be reached adding respectively reciprocity, removing ethnocentrism or doing both actions. If the starting situation is an HE, then by adding reciprocity an IE may be reached, while from IEE, by removing ethnocentrism an IE may be obtained. The reverse processes may not be done so that, once that integration is obtained, it is impossible to create segregation from it only by changing the model parametrization.

Until now we have shown sufficient conditions for convergence. With the next corollary we provide necessary conditions for convergence to the HE-HEE class of equilibria, if every agent of every ethnic group uses the same socialization scheme, that we have called *'symmetric'*:

Corollary 2: *A Necessary condition for convergence to a HEE or HE with symmetric socialization schemes is that reciprocity does not enter in the socialization schemes.*

Corollary 2 states that an ethnic hierarchy may be substained in the long run only if no reciprocity holds. This necessary condition may be of some relavance since, in some political talkings on immigration, reciprocity is viewed as a way to introduce incentive for the building of a good attitude world. Sometimes, the subtle justification for these action calls, is in the willingness of maintaining the present ethnic social ranking. With this framework we show that both these reasonings may be wrong since reciprocity is the principal scheme for allowing cross-dependence of cultural values and thus for integration, as defined here. On the other side reciprocity, if applied in this symmetric socialization scheme, is incompatible with the preservance of an ethnic social hierarchy.

4.4 Optimal Socialization Effort and Welfare Analysis

In the previous part of the paper we have analysed the long run properties of these dynamics. However, the crucial choice that parents perform in this process is the socialization effort: it it thus interesting to see what happens to this effort during the convergence process. In particular it is of some relevance to study what happens to optimal effort if there is majority and a minority, since there are close relationships with the situation of immigrant in some countries. Moreover, we study the relationship between the optimal effort and the change in values, meaning the way in which the two groups integrate. Then we see how to measure parents' frustration and if there is some relationship between the effort, the change in values and te frustration. We thus build up a two ethnic groups framework in order to understand the basic properties of this choice.
Suppose to have 2 ethnic groups, i and j. Suppose that they act with reciprocity and ethnocentrism so that the only attitudes involved in the dynamics are V_t^{ij} and V_t^{ji}. Suppose that oblique socialization weights are given by population shares, as in equation (4). Suppose that population shares do not change in time so that $p_t^i = p_{t+1}^i, \forall t$: the case for changes in the population shares will be addressed in section 6 in which the analysis of dynamic weights will be performed. Suppose that the cost function is $c(\tau_t^k) = (\tau_t^k)^2$ for $k = i, j$. Moreover, define $\Delta_t^i = (V_t^{ij} - V_{t+1}^{ij})^2$, $\Delta_t^j = (V_t^{ji} - V_{t+1}^{ji})^2$ and, from the utility function specification, define the frustration $F_t^i = \Delta_t^i + c(\tau_t^i)$, and the same definition for F_t^k. The first indicates the loss of the parents and, contemporarily, is a measure of the change in values between the two generations, while the second indicate the total frustration frustration of the parents since it is the sum of the loss and the socialization costs.
We now start studying some conditions under which these two groups differ in terms of levels of these measures. We can thus state the following:

Proposition 4. *A necessary and sufficient condition for $\tau_t^{i*} > \tau_t^{j*}$, $\Delta_t^i > \Delta_t^j$ and for $F_t^i > F_t^j$ is that $p_t^i < \frac{1}{2}$.*

Proof. See Appendix B □

The first part of the proposition simply states that minority groups tend always to socialize more, and this is in line with what found by Bisin and Verdier (2000, 2001). In fact in this two groups model minority values have a smaller weight than majority values in the determination of the oblique socialization effect, and thus minority parents tend to socialize their children more in order to reduce the loss. Is this higher effort effective in order to reduce the integration of minority children? Said differently, given $\tau_t^{i*} > \tau_t^{j*}$ are minority children moving slower than majority children towards integration? The second part of the proposition states that $\Delta_t^i > \Delta_t^j$, meaning that even if minority parents

put a high effort in the socialization, their chidren have values moving faster than the majority and, in the mean time, minority parents experience a higher loss. The third part of the proposition states that, as a consequence of the previous two, minority parents always experience a higher frustration than majority parents.
Call $D_t^i = D_t^j = D_t = (V_t^{ij} - V_t^{ji})^2$, representing the differences in the values of the two groups. It is now interesting to analyze how τ_t^{i*}, Δ_t^i and F_t^i changes with p_t^i and D_t^i.
The values at optimum are: $\tau_t^{i*} = \frac{p_t^{j\,2} D_t}{1+p_t^{j\,2} D_t}$, $\Delta_t^i = \frac{p_t^{j\,2} D_t}{(1+p_t^{j\,2} D_t)^2}$ and $F_t^i = \frac{p_t^{j\,2} D_t}{1+p_t^{j\,2} D_t}$. The first interesting thing to notice is that at optimum it always happens that $\tau_t^{i*} = F_t^i$, meaning that the effort level coincide with the frustration level, and thus they have the same properties. It is thus enough to study the effort in order to induce the frustration properties. We first study the effort properties, and thus the frustration properties: we have that, in equilibrium, $\frac{\partial \tau_t^i}{\partial p_t^j} > 0$, since, as we previously said, higher opponent population means higher effort and higher frustration. Then we have that $\frac{\partial^2 \tau_t^i}{\partial^2 p_t^j} > 0$ if and only if $p_t^j \in [0, \frac{1}{D_t \sqrt{3}}]$. In figure 2 this case is represented by the area below the upper line. This means that generally an increase in opponent's population share has an increasing effect. This effect is decreasing only for high p_t^j and high D_t, that menas only if i are strict minority with values very different from the majority. However, since $\dot{D}_t < 0$, permanence in the upper-right part of the graph may be only temporary, and thus it will be more likely to be in the case in which $\frac{\partial^2 \tau_t^i}{\partial^2 p_t^j} > 0$.
Turning now to the loss analysis we have that $\frac{\partial \Delta_t^i}{\partial p_t^j} > 0$ meaning that, if the oponents' share increase, then i loss increase. This is again due to the fact that the higher effort does not counterbalance the effect of the shift of the oblique socialization effect. In the same time this also means that higher opponents' effort means that i children move faster towards integration. If we then look at how this happen, we have that $\frac{\partial^2 \Delta_t^i}{\partial^2 p_t^j} < 0$ if and only if $p_t^j \in [\sqrt{\frac{4-\sqrt{13}}{3D_t}}, 1]$. In figure 2 this is represented by the bottom curve. Notice that this never happens if $p_t^j < \sqrt{\frac{4-\sqrt{13}}{3}}$, represented by the straight line. Then, if $p_t^j \in [\sqrt{\frac{4-\sqrt{13}}{3D_t}}, \frac{1}{D_t\sqrt{3}}]$, we have that $\frac{\partial^2 \tau_t^i}{\partial^2 p_t^j} > 0$ and $\frac{\partial^2 \Delta_t^i}{\partial^2 p_t^j} < 0$, represented by cases of middle-high levels of D_t or of p_t^j.

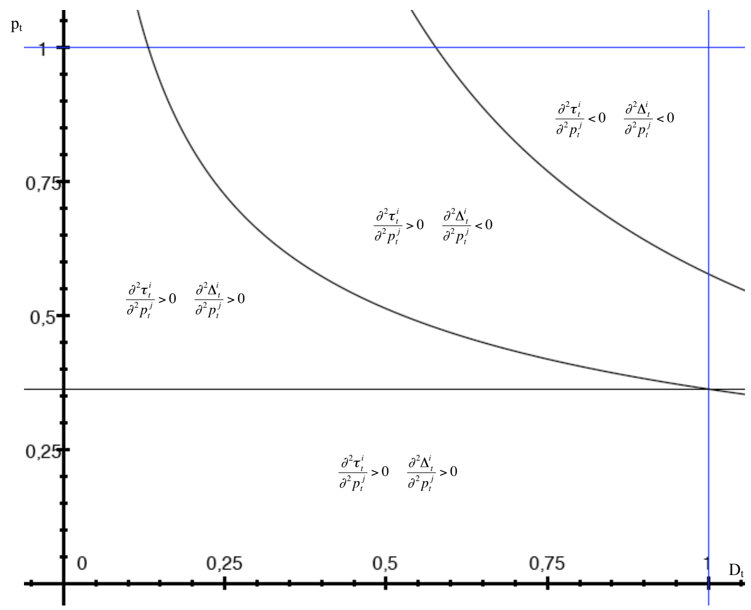

Figure 2: Comparative Statics

Suppose now to have 3 worlds in which i is a strict minority with the same population shares, but in which the level of distance in values between the two groups differs: High, Middle and Low. so that a High-Distance, Middle-Distance and Low-Distance world are identified. Then we can say that:

Corollary 3: *Suppose i to be a strict minority under demographic pressure. Then the increase in minority socialization effort and parents' frustration is higher in medium-high distance world, while parents' loss and children' values change is higher in a medium-low Distance. With respect to medium distance world, high and low distance worlds experience lower changes in all the quantities.*

Last corollary states that very integrated and non integrated minorities resists relatively better to demographical pressure than minorities on a medium integration path. We can thus say that minorities half in the way of integration react in such a way that their change in values, parents' frustration and socialization effort is maximum. This tells us that ethnic groups that started their integration process and they are on the way of being high enough integrated are the most frustrated and thus the ones that can present more problems to the majority since they can show much more resistence to this integration process. Even if we do not predict that this minorities tend to invert the integration process (impossible under this modellization), this resistence effect goes in this direction. Non integrated groups, on the other side, since they change less in values, and thus have less frustration levels, may seem unlikely to overreact to the integration process. In the same way way integrated minorities show their weakness in term of maintaining their values and their bigger integration potential. Looking at the graph we can also say that large majorities under demographic pressure always show increasing effects of demographic pressure, whatever the distance between groups. On the other side, taking the case in which $p = 0.5$, then the highest level of values change occurs for a middle-distance environment, close to $D_t = 0.5$
Looking now at the effect of different population shares we can say that:

Corollary 4: *Suppose to live in a High-Distance world. Then the increase in socialization effort and parents' frustration is maximum if the group happens to be a minority, while parents' loss and children' values change is the highes if the group is a majority. If population shares are of comparable size then the increase in parents' loss and children' values change is lower than for the case of a large majority, while parents' effort and frustration is higher than for a strict minority.*

The last corollary states that, in a diverse environment, if a majority is under demographic pressure, then its changes in values are at its maximum, while changes in parents' frustration are maxima if the groups is a minority. Thus, even if we have previously proved, in proposition 4, that minorities have the maximum values between the two groups, this analysis shows that majorities may experience a higher reaction to population shocks, being a crucial element for the understanding of the dynamics.
We now turn to the analysis of the effects of a shock in the cultural distance D_t. In this case the effects are much more clear since we can say that $\frac{\partial \tau_t^i}{\partial D_t} > 0$, $\frac{\partial^2 \tau_t^i}{\partial^2 D_t} < 0$, $\frac{\partial \Delta_t^i}{\partial D_t} > 0$, $\frac{\partial^2 \Delta_t^i}{\partial^2 D_t} < 0$, meaning that independently from the population shares and the distance the effect of an increase in the distance of values have always the effect of increase effort, frustration, loss and values change, but in a decreasing way.
All this analysis is of some interest if we want to analyse what happens to attitudes in a society of natives N when immigration of an ethnic group I happens. We consider the case in which immigrants represents a minority. Then we have that, as far as immigrants remains a minority, then immingrants socialize more than natives. However, this higher attention on segregation does not mean that their actual integration is lower than the native one, since it always happen that $\Delta_t^I > \Delta_t^N$, meaning that immigrants cultural values change always faster that Natives, both in absolute and in relative terms. Howver this brings I parents to be much more frustrated that N parents, and this may be of some importance for policy design. Moreover, as we last pointed out, if there is a shock in the cultural distance it always happens that I and N respond increasingly in this shock with a higher effort, higher loss but, in the same time, bigger steps towards integration. It is then interesting to see what happens if immigrants' population suddenly increase: even if the analysis of what happens in the dynamics is analysed in section 6, we can still the differences in responce among societies, withe the help of figure 2 (since we are now considering an increase in own group population shares the signs of the derivative must be inverted). In particular we can say that if I and N are very distant then the decrease in effort and frustration is lower than for the case of I and N very close again underlining the higher effor of segregation of these groups. Middle distance ethnic groups, on the other hand, show that frustration increase less

that segregated minorities and integration increase more that integrated minorities.

Until now we have supposed that parents knows perfectly the oblique socialization weights and thus may optimally choose the socialization effort. However this is not always the case. If we consider the case of immigrants it may be the case that parents overestimate the impact of opponent groups in the oblique socialization of own children and thus oversocialize children. Suppose $\tilde{p}_t^j$ be the perception i parents have of j share. Then we have that: $\tilde{\tau}_t^{i*} - \tau_t^{i*} = \frac{D_t(\tilde{p}_t^{j\,2} - p_t^{j\,2})}{[1+D_t\tilde{p}_t^{j\,2}][1+D_t p_t^{j\,2}]}$. The properties of $\tilde{\tau}_t^{i*}$ are the same as the ones studied for a demographical shock.
It is then far more interesting the effect on the dynamics of values and on the frustration. In this case we have:
$\Delta_t^i = \frac{p_t^{j\,2} D_t}{[1+\tilde{p}_t^{j\,2} D_t]^2}$ and $F_t^i = \frac{D_t(\tilde{p}_t^{j\,2} + p_t^{j\,2})}{[1+\tilde{p}_t^{j\,2} D_t]^2}$
We thus have $\frac{\partial \Delta_t^i}{\partial \epsilon_t^j} < 0$ and $\frac{\partial F_t^i}{\partial \epsilon_t^j} > 0$. Thus the overestimation of opponents' share bring to a lower change in children values. However, since the socialization effort is higher the greater cost offset the last advantage and the frustration increases. Since this results holds even when $\epsilon_t^j < 0$ then we have that the opposite hold with underestimation.
Thus we can state that:

Corollary 5: *Any group that overestimate the opponents' share tend to socialize more, to change less in values and to be more frustrated than in case of perfect population estimation. If a policymaker convince each group that the opponents' share is lower than in reality, then integration of values speed up and agents frustration is lowered.*

5 Asymmetric Socialization Rules

In the previous sections we have analysed the cases in which every agent of every ethnic group follows the same socialization scheme, so that symmetric socialization rules are implied. In particular every ethnic group applies a reciprocity schemes towards any other groups, or no one does towards anyother. Moreover everyone is ethnocentric or no one is ethnocentric. Emulation Rule imposes that $w_{ij}^{kj} > 0, \forall i, j, k, t$ so that all ethnic groups are considered in the process and every ethnic groups considers all the other groups in the socialization scheme. These cases, however, can limit the analysis since different ethnic groups may show different socialization schemes depending on various social situations, or simply for any reason that we can think causes heterogeneity in socialization schemes: in particular it can be that a given group i can consider j's cultural traits as relevant while k's traits as irrelevant and thus using an appropriate socialization scheme is relevant.
In order to analyse these situations, we introduce a notation borrowed from network theory. We use this kind of setup since, on one side, there is a strong link between the network structure and the transition matrices we use for proving convergence since a matrix is irreducible if and only if the associated directed graph is strongly connected; on the other side networks may give a more intuitive view of the relationships between ethnic groups in attitudes formation schemes, and thus it will be easier to identify relationships among ethnic groups and oblique socialization structures. Moreover, in order to study steady state classes, there is no need to know the weights intensity, but just if they exists or not. In fact, proposition 3 states that if a given oblique socialization structure exists, then convergence to a particular steady state happens without regards to the the intensity of the single influences. Thus, we are interested in wether the links exist or not, rather than their intensities. The following analysis is also linked with De Marzo et al (2003): in their Appendix 1.C they prove convergence points for the case of non-strongly connected matrices. The cases presented here are thus related to them, with the exeption of the fact that, as previously argued, in this case agents are allowed to have group dependent τ_t^{i*}. These cases are thus a way in order to incorporate their aalysis in this context.
Suppose each V_t^{ij} is a node, and call U the set of all the nodes.
Then, the directional link $V_t^{ij} \to V_t^{kw}$ is built if and only if $w_{ij}^{kw} > 0$.
Call P_{ij}^{kw} the set of the possible paths, both direct or non-direct, from V_t^{ij} to V_t^{kw}
Define now a *sink* the set
$S \subset U$: $S \equiv \{V_t^{ij} : P_{ij}^{kw} = \emptyset, P_{ij}^{nx} \neq \emptyset, \forall V_t^{ij} \in S, \forall V_t^{nx} \in S, \forall V_t^{kw} \notin S\}$.
Thus a sink is a set of nodes such that there is no path from any of them to any node outside the sink. The sink may be composed either of only one node or of more than one node. In the first case a node V_t^{ij} is a sink if and only if

$w_{ij}^{ij} = 1$ since in this case any other $w_{ij}^{kw} = 0$ and thus no links are formed towards outside. In terms of our model this means that the attitude is not questioned, and thus no dynamics will be shown for this trait. As a consequence, if ethnocentrism applies then V_t^{ii} is a sink. In the case in which the sink is composed of multiple nodes, then they are strongly connected, meaning that if $V_t^{ij} \in S, V_t^{nx} \in S$, then $P_{ij}^{nx} \neq \emptyset$. Moreover, taken any node not belonging to any sink, there should exist a path that connect it to a sink, otherwise it would belong to a sink itself. This means that if there is only one sink then there should exist a path from any element out of the sink to an element of the sink.
With this framework we have that, depending on the relevance parameters, the structure of the network may differ but, given that they are time independent, the structure of the network does not change with time.
In order to control for asymmetric socialization rules, we start relaxing a little bit the Emulation Rule as previously defined. In that case $w_{ij}^{kj} > 0 \forall i, j, k$, meaning that all agents consider as relevant all the other ethnicities in the society in their socialization scheme. A more complex relational structures with some $w_{ij}^{kj} = 0$ may be considered. Thus:

Assumption 2: Consider a Socialization rule such that $w_{ij}^{kw} = 0, \forall w \neq j$, and $w_{ij}^{kj} \geq 0, \forall k$.

Note that this is compatible with $w_{ij}^{ij} = 0$, for some i, j, so that the diagonal entries of the weights, but the diagonal of the transmission matrix is positive since, by assumption 1, $\bar{\tau} > 0$. Then the following proposition holds:

Proposition 5. *A sufficient condition in order to have an HE is that*

- *Assumptions 1 and 2 hold*
- *The nodes V_t^{ij} form a single component $\forall i, t$*
- *each component is strongly connected or has only one sink*

Proof. See Appendix B. □

Proposition 4 states that HE may be reached under a big variety of socialization structures as far as all the $V_t^{ij}, \forall i$, depend directly or indirectly only from each others (this is the case in which the component is strongly connected) or there is one sink, so that every other depends on it. Figure 3 gives some examples of these cases and in particular of strongly connected networks.

What does this mean in terms of relationships among groups? The first implication is that is it not needed that

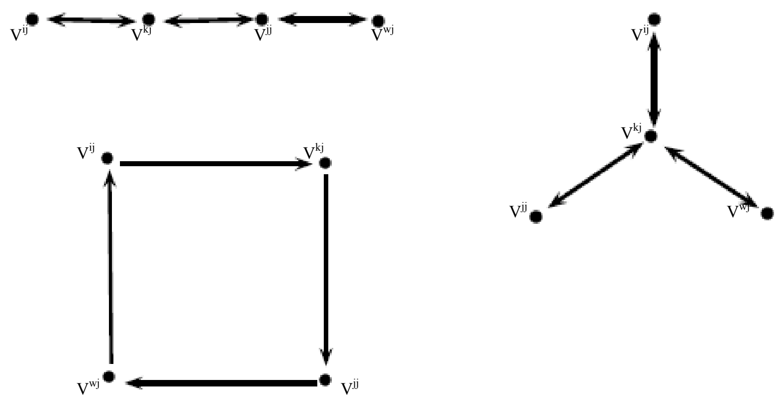

Figure 3: Strongly Connected Network

every group has contacts with all the other groups or consider them reliable during the socialization process in order to have a hierarchy of attitudes. Consequently we can have convergence to a HE even if there are strong frictions in the contacts among groups. The idea is that it is not necessary to be in touch with group k in order to know and maybe take its values as ours. It could be enough to be in touch with group j that is in touch with group k so that, by means of j we can consider k values in our socialization process. Consider, for example, the first graph of figure 3. In this case i and k reciprocally get influenced and j and w do the same. However k and j are also reciprocally linked. These two ethnicities can be considered as cultural bridges for ethnic groups that do not have contacts. Suppose for example that i and j are ethnicities that refuse to get influenced each other, while both of them have contacts with k. This may be the case for two conflictual ethnic minorities (i and j) and a majority (k). The role of the majority in this case is not to simply report to i the j's values and viceversa, but to incorporate these values in its values and, by this process, making them acceptable by the third ethnicity. i does not trust j, but trusts k so that j's values may become acceptable if proposed by k, after having internalized them through its socialization process. However, the case for these *cultural bridges* may only be one case.

The second case reported by the graph is the one in which one ethnicity is considered as a *cultural hub*. Suppose that an ethnic group k, for its role in the society, is the most open ethnic group such that k children have contacts with all the other groups and gets influenced by them and, also, all other groups' children get influenced by it. In this case the 'hub' k is a collector of all others' cultural values, it produces a synthesis and influences the others. In this way everyone gets everyothers' values by means of the cultural hub, so that the weight vector that k uses become crucial for the determination of the steady states values.

The last case represented in figure 3 happens if there are '*cultural circles*'. In this case no groups has a predominant role but it just processes a little part of the overall cultural values and it passes to other groups in a circle. This last cases also makes clear the role of vertical socialization in ensuring convergence. Suppose that a scheme as the one with a cultural circle holds. Suppone then that parents do not socialize at all their children so that $\tau_t^i = 0, \forall i, t$, so that assumption 1 does not hold. Then in this case convergence does not happen since there are cyclic matrices and thus a fluctuation of cultural traits is shown, unless all the values happen to coincide at time 0.

Until now we have analysed the case in which a HE may be reached. We can similarly consider the conditions for reaching a IE.

Proposition 6. *A sufficient condition in order to have an IE is that*

- *Assumptions 1-2 and Oblique Socialization Stability holds*
- *The nodes V_t^{ij} form a single component $\forall i, j, t$*
- *the component is strongly connected or has only one sink*

Proof. See Appendix B. □

This proposition is very similar to the one for HE with the difference that now all cultural values may be somehow linked each other. Until now, the instrument that makes this possible is Reciprocity. However, it is not needed that everyone uses reciprocty towards anyother in order to obtain an IE. Suppose, in fact, that every ethnic group uses a Emulation Rule (resp. Ethnocentrism Rule) so that a HE (resp. HEE) is reached. Suppose now that one group starts to use reciprocity towards any other group. Figure 4 provides a graphical example for this case.

In this case group k uses reciprocity towards anyother. As a result, the long run equilibrium will be an IE in which the final attitude of everyone towards anyother is given by the attitude that everyone had towards k at the beginning, since the set $S = \{V_t^{ij}, \forall i, t\}$ is a sink. In this way, the role of reciprocity is much more clear: if reciprocity is used by an ethnicity everyone think is bad, then a bad attitude of everyone towards anyother may be a result. If, on the reverse, it is used by a well reputed ethnicity, then a long run equilibrium in which everyone have good attitudes towards other may be likely to be observed.

Another interesting case happens when reciprocity chains are observed, meaning that each ethnic group uses reciprocity towards another one and a chain or a circle is observed, so that again an IE is reached.

In the case for IE, as in the HE cases, the roles for cultural bridges and cultural hubs may be equally reproduced. Moreover, proposition 5 opens the rowad to different schemes of oblique socialization others that reciprocity such that,

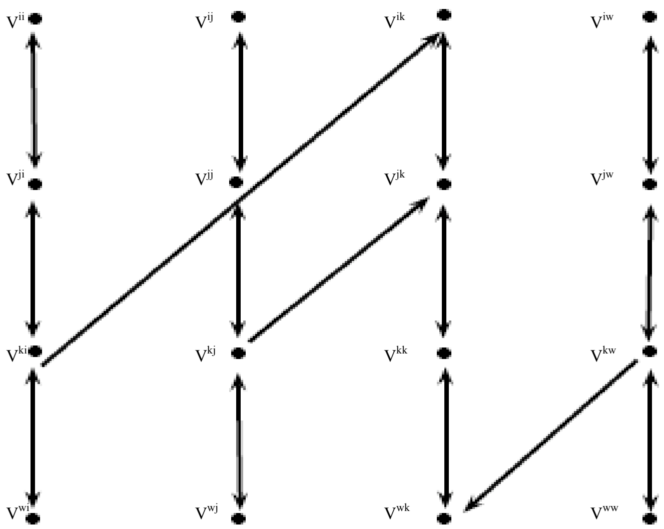

Figure 4: Reciprocity

linking all nodes together, may be responsible for convergence to an IE.
Until now we provided sufficient condition for convergence to the different classes of equilibria given asymmetric socialization rules. We study now the existence of some necessary conditions under this asymmetric socialization schemes framework. A condition for both hierarchy and integration classes of equilibria is the following:

Corollary 6: *A Necessary condition for convergence to a HE and HEE is that all the sinks belonging to a given component converge to the same value.*
A Necessary condition for convergence to an IE and IEE, is that all the sinks converge to the same value.

This corollary states that if there are more cultural models influencing the same cultural traits, then an equilibrium belonging to one of the previous described classes cannot been shown. In fact, every trait is differently influenced by the multiple models and thus every trait converges to a different value. Now, if a multiplicity of different cultural models may be compatible with a convergence to a HE, since every component may have one different sink, on the contrary in order to have an IE it is necessary that all the cultural models of the society converge to the same value. Thus a multiplicity and diversity of cultural models may preclude long run integration equilibria.

6 Time-Dependent Oblique Socialization

Starting from equation (6) we have constrained the oblique socialization rule to be fixed along time. However, this may not be the case, since the society composition may change, and thus weights may change as well along time following different possible rules. We thus now study what happens if a more general specification of the cultural dynamics is taken into account, thus reconsidering the formulation in equation (5):

$$V_{t+1}^{ij} = \tau_t^i V_t^{ij} + (1 - \tau_t^i)(\sum_{k,w} w_{t,ij}^{kw} V_t^{kw})$$

so that oblique socialization weights may change with time. It is not in the purposes of this paper to produce specific dynamics for these weights: consequently, we analyse sufficient conditions in order to get convergence with time-dependent weights, and sufficient condition for convergence to particular classes of steady states, independently from the specific weights dynamics we consider.

Assumption 3: *(Symmetry)* There is a symmetric oblique socialization rule if $\exists T : w^{kw}_{t,ij} \neq 0 \Leftrightarrow w^{ij}_{t,kw} \neq 0, \forall i, j, k, w, \forall t > 0$.

This assumption is satisfied if, at least after some point in time, if a cultural trait A directly influences cultural trait B, then the trait B influence also the trait A. Speaking with network language this means that all the links that exist in a network, after a period of time T have to be bidirectional so that the respective matrices are symmetric. To notice that a direct consequence of this assumption is that every component of the derived directed graph is strongly connected.

Assumption 4: *(Temporal Stability)* There is a temporal stable oblique socialziation rule if $\exists T : w^{kw}_{T+t,ij} \neq 0 \Leftrightarrow w^{kw}_{T+t+1,ij} \neq 0$ or $w^{kw}_{T+t,ij} = 0 \Leftrightarrow w^{kw}_{T+t+1,ij} = 0, \forall i, j, k, w, \forall t > 0$.

Thus Oblique Socialization Stability is a property of weights such that, after some periods of time, the way in which ethnicities are influeced each others is stable. Namely, if i agents do not consider j agents, they continue with this scheme forever and if they consider them they continue in this way forever. As a direct consequence of this property $w^{kw}_{T+t,ij} = 1 \Leftrightarrow w^{kw}_{T+t+1,ij} = 1$ and $w^{kw}_{T+t,ij} \in (0, 1) \Leftrightarrow w^{kw}_{T+t+1,ij} \in (0, 1)$. Moreover, if assumption 4 holds, then after time T the network structure is fixed.
Thus, given these assumptions, we can state the following:

Proposition 7. *If Assumptions 1-3-4 hold, then any* GCSF dynamics *converges to a steady state. If Assumptions 1-4 hold and, at time T there exists only one sink for each component, then convergence happens.*

Proof. See Appendix B. □

Last proposition states that if we set up any weights dynamics such that symmetry is satisfied and after a time T it is also stable, then convergence happens. An example for this happens if we consider a dynamics as the one represented in equation (4) that we report down here:

$$V^{ij}_{t+1} = \tau^i_t V^{ij}_t + (1 - \tau^i_t)(\sum_k p^k_t V^{kj}_t)$$

so that the socialization weights are represented by the population weights. As previously argued, this represents the most frictionless society we can imagine. Suppose then that the population dynamics is such that no group ever gets extinguished. Then we can state the following:

Corollary 7: *If cultural dynamics is represented by equation (4) and no population ever get extinguished, then convergence to an HE occurs. If reciprocity is introduced, then convergence to an IE occurs.*

Last corollary put into this context what Brueckner and Smirnov (2007, 2008) found in their works, and shows how this is only one specific case that can be represented in this cultural evolution context.
From a technical point of view, the second part of proposition 6 also extends the results of Brueckner and Smirnov (2007, 2008), since we add sufficient conditions for convergence if the transition matrix cannot be rewritte as a block diagonal matrix with irreducible diagonal blocks, as shown in the proof of proposition 6.
Then, we can also state the following more general sufficient condition for convergence to a HE:

Corollary 8: *If assumptions 1-2-3-4 hold, then, if at time T the network derived from the socialization weights has only one sink per component, then an HE occurs.*

Last corollary states that, apart from the sufficient conditions for convergence, since assumption 4 implies time stability of network structure, the structure of the network at time T can indicate the class of steady state it will be reached.

6.1 Optimal Socialization Effort Dynamics

Corollary 5 introduces the case of change in population shares within the rule we used before in order to study the effort properties and its effects on change of values and frustration. In that case only comparative statics analysis was performed: here we complete that analysis by showing what happens during the convergence process. Moreover we address here a second problem: from the simulations in symmetric cases it can be seen that all socialization efforts converges to 0 and that they do it monotonically. This monotonicity is due to three main reasons: there is asymmetry, there is no change in the structure of the socialization and thus there are no break in the paths towards convergence, and the population shares vector does not show any dynamics or change. Apart from the first condition, the break of the second condition can be observed when, in Appendix A, we provide endogenous weights with the possibility of breaks in the socialization structure, so that monotonicity of socialization efforts is not shown. The third case can be analysed here. The setting is similar to the one presented above: suppose to have only 2 groups, Natives (N) and Immigrants (I). Suppose, for simplicity, that they act with ethnocentrism and reciprocity, with weights proportional to the population share. Suppose then that there is a constant inflow of Immigrant in the society such that $p^I_{t+1} > p^I_t$. We can also think at a no immigration but at a higher fertility rate of immigrants ($n^I > n^N$) such that the dynamics of population can be derived as $p^N_{t+1} = p^N_t + p^N_t(1-p^N_t)(n^N - n^I)$. From equation (3) we can thus write the objective function of N parents

$V^* - (1-\tau^N_t)^2(1-p^N_t)D_t - c(\tau^i_t)$

and thus, in equilibrium we have that $\frac{\partial \tau^{N*}_t}{\partial p^N_t} < 0$ and $\frac{\partial \tau^{N*}_t}{\partial D_t} > 0$. The same happens for the immigrants. Given the definition of the socialization structures, we can state that it always happens that $D_t \geq D_{t+1}$ since convergence of attitudes always happens, since, at each period, values involved in the cultural transmission converge towards a weighted mean.

Now, given $\frac{\partial \tau^{N*}_t}{\partial p^N_t} < 0$ and $\frac{\partial \tau^{N*}_t}{\partial D_t} > 0$, consider first the case for immigrants: since $D_t \geq D_{t+1}$ and $p^I_t < p^I_{t+1}$, both forces act in order to reduce their socializaiton effort and thus it happens that $\tau^{I*}_t > \tau^I_{t+1}{}^*$. In fact, their increasing population share makes the oblique socialization more biased towards their own values.

Consider now the Native population: since $D_t \geq D_{t+1}$ and $p^N_t > p^N_{t+1}$, the two forces have conflictual effects on the optimal socialization effort, and thus it can happen that $\tau^{N*}_t < \tau^N_{t+1}{}^*$, and thus non monotonicity may be observed. In fact if convergence of attitudes reduces the parents' loss, on the other side the reduction in population shares makes the oblique socialization more favourable to the other ethnic group and this has a positive effect on the optimal socialization choice.

Intuitively we can say that if the population change is faster than the convergence in attitudes, then natives will tend to produce more socialization effort. We now provide an esemplificative example.

Suppose to use the standard cost function $c(\tau^i_t) = \tau^{i2}_t$. Then we have that:

$\tau^N_t = \frac{(1-p^N_t)^2 D_t}{1+(1-p^N_t)^2 D_t}$

$\tau^N_{t+1} = \frac{(1-p^N_{t+1})^2 D_{t+1}}{1+(1-p^N_{t+1})^2 D_{t+1}}$

and thus $\tau^N_{t+1} > \tau^N_t$ if and only if $\frac{D_t}{D_{t+1}} < \frac{(1-p^N_{t+1})}{(1-p^N_t)}$.

Corollary 9: *If the rate of growth of I population has been higher than the rate of convergence of attitudes, then an increase in N socialization effort has happened.*

However this can only be an ex-post decription since the rate of convergence depends on optimal efforts, and until now we are not able to provide thresholds for the sole fertility rates such that this happens.
This however may have some impact on policy description since, in this simplified framework, immigrants may start with higher socialization efforts given their minority status, but their effort is always declining. On the other side, natives, starting from low levels of socialization, may choose to produce more effort if there is a great imbalance between integration of values and population change.
Moreover, in this framework parents use actual population shares in order to forecast chidlren' oblique socialization. This does not happen in real world, and parents use some expected values that can be biased, especially if fear of immigration holds, tending thus to overvaluate the impact of imigrants on the oblique socialzation. Thus the effect may be more complex and needs a deeper analysis.
We now present a numerical simulation in order to make this result clear. The initial attitudes matrix is the same as in the previous cases, and the initial population vector is $p_0^i = 0.85$, $p_0^j = 0.05$ and $p_0^k = 0.1$. Then we set up teh following fertility rates (number of children per parent): $n^i = 2$, $n^j = 3$ and $n^k = 2.5$. In this way the dynamics of i population is given by:

$$p_{t+1}^i = p_t^i + p_t^i\left[(1 - p_t^i)(n^i - n^k) - p_t^j(n^j - n^k)\right]$$

and the dynamics for the other two groups may be derived in a similar way. We thus wanted to capture the idea of a society initially composed of a majority i, but in which migration of two groups happen. Both of them have higher fertility rates than the natives. Given the simplicity of this dynamics, at the end the group with the highest fertility rate will invade the society. In figure (5) are represented the simulations for the attitudes in the first graph and for population shares (lines) and socialization efforts (dots) in the second graph.

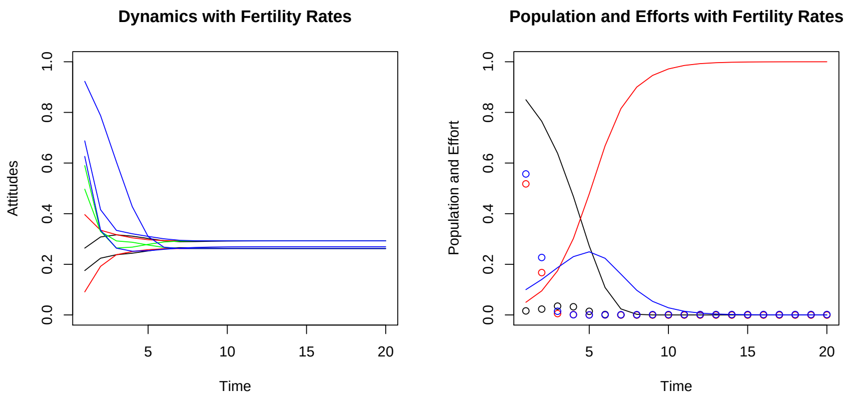

Figure 5: Simulations with Fertility Rates

A look at the first graph shows that, even if reciprocity is used, at the end no integration equilibrium happens. However this is simply due to the fact that, at some point, some groups gets extincted and thus they do not enter anymore in the socialization process of others and thus the integration of values cannot happen. In the second graph we can see that, at the end, only the group with the highest fertility rate survives, while the other two are extincted. In particular the second migrant group experience ant the beginning a growth of its population shares, but then, due to the fertility differential with the other immigrant group, it gets extincted as well. If we then look at the socialization efforts we see that one group (the majority i) experience an initial growth in the socialization effort and then a decline. This is due to the fact that at the beginnig the convergence of attitudes is relatively slower than the reduction of i population, and thus i parents, in order to compensate for that, decide to produce an higher effort. When then the rate of population change declines, then the opposite happens and convergence towards the zero level happens.

7 Endogenous Oblique Socializations

In the previous sections we have shown how the society will end up with a particular type of steady state given different weights structures. However, if real societies are analysed, we should provide a criterion in order to decide which weight structure is more suitable to the analysis in order not to give it exogenously.

In equation (5) we provided a general dynamic for which any V_{t+1}^{ij} could potentially depend on any V_t^{kw}. Here we consider simpler dynamics derived from the ones described by the 4 socialization rules previously defined. Dynamics in this context differ in relation to the structure of Oblique Socialization. In particular, given ethnicity i, oblique socialization crucially depends on what extent agents i trusts other ethnicities in making their judgments or how much the degree of similarity among them influences the attitude formation scheme. We thus consider the case in which the weights agents assign to other ethnicities' judgments only depend on a measure of similarity of values.

In order to study this kind of endogeneity, we introduce the concept of *Cultural Similarity*, s^{ij}, being a measure of how a given ethnicity i is close to another one j. This similarity will thus have an impact on the weights accordingly to the following basic rules:

- $w_{t,ij}^{kj} > 0 \Leftrightarrow s_t^{ik} > 0$
- $w_{t,ij}^{kj} = 0 \Leftrightarrow s_t^{ik} = 0$
- $w_{t,ij}^{kw} > w_{t+1,ij}^{kw} \Leftrightarrow s_t^{ik} > s_{t+1}^{ik}$

These properties state that the weight is positive if and only if the similarity is positive so that two dissimilar groups do not interact in the attitude formation scheme. Then the weight is increasing in the similarity between the considered ethnicities. Note that $w_{t,ij}^{kj}$ is independent from j so that $w_{t,ij}^{kj} = w_{t,in}^{kn} = w_{t,ik}^{kk}$ and $w_{t,ii}^{ji} = w_{t,ik}^{jk}$ so that ethnocentrism does not hold in this specification.

How now to determine the similarity? The measure of the similarity can be done on proper sociological studies given the particular environment we would like the model to be applied to, but this is not the road we would take in this theoretical work. Thus, coherently with the model in which agents only cares about ethnicity, similar type vectors means that they show similar cultural values, thus we propose an endogenous similarity index.

Define $\Delta_t^{ij} = f(V_t^i, V_t^j) : [0,1]^{n^2} \mapsto [0,1]$ such that $\frac{\partial \Delta_t^{ij}}{\partial |V_t^{ik} - V_t^{jk}|} < 0 \; \forall k$ and $\Delta_t^{ii} \leq \Delta_t^{ij} \; \forall i, j$.

This element is a measure of cultural values distance among ethnic groups.

Define $s_t^{ij} = s(\Delta_t^{ij}) : [0,1] \mapsto [0,1]$ such that

- $s_t^{ii} > 0$;
- if $\Delta_t^{ij} > \Delta_t^{ik} \Rightarrow s_t^{ik} > s_t^{ij}$
- $\exists \bar{\Delta}$:s$(\Delta_t^{ij}) > 0 \; \forall \Delta_t^{ij} \leq \bar{\Delta}$, and $s(\Delta_t^{ij}) = 0 \; \forall \Delta_t^{ij} > \bar{\Delta}$, with $\bar{\Delta} \in [0,1]$.

This element is a similarity function such that it is decreasing in the value distance and self-similarity is always positive. The third condition states that there could exist a threshold under which the similarity is set as 0, and that if $\Delta_t^{ij} = 0$, meaning that the two ethnicities have identical values, then their similarity has to be positive also in the case in which $\bar{\Delta} = 0$. We call $\bar{\Delta}$ *Openness Propensity.* In fact, for high levels of $\bar{\Delta}$, the agents consider also far ethnicities in their socialization schemes, so that they are open towards big changes in their values. The opposite happens for low levels of $\bar{\Delta}$.

This rule is similar to Golub and Jackson (2009) in which they analyse the case of convergence of opinions in presence of homophily, measured as the willingness to comunicate with closest people using an euclidean metric in order to compute the distance. In their case, however, since there was no parameter such as $\bar{\Delta}$ convergence to a common value always happen, and the analysis is focused on the speed of convergence. Thus our analysis is mainly focused on the openness parameter and its effect on convergence, with the usual additional biased due to the parents' socialization effort.

For the time being we have that all groups use the same similarity function and the threshold is not ethnic specific, thus we have that $s_t^{ij} = s_t^{ji}$.

Definition 3: Call *Basic Cultural Distance* a distance such that $\Delta_t^{ij} = \sum_k x^k |V_t^{ik} - V_t^{jk}|$ with $\sum_k x^k = 1$ and $0 \leq x^k \leq 1 \; \forall k$, and where there is a group independent similarity function $s(\Delta_t^{ij})$, and a group independent parameter $\bar{\Delta}$.

This is the most simple cultural distance we can think about. In particular the distance between i and j is defined as a weighted mean of the absolute value of the differences of all their entries. Moreover we assume that all agents use the same similarity function and that they have the same openness propensity (we extend to heterogenous propesities later in this section). With this specification we have that $s_t^{ij} = s_t^{ji}$ so that, taken two ethnic groups, they agree on the degree of similarity between them. We then analyse what happens if this openness parameter changes with groups. The next analysis is devoted to these two cases.

7.1 Group Independent $\bar{\Delta}$

Before starting to study the set of all attitudes we restrict the analysis to a more simplified case.
The model is the same as before with the only exeption that each group i has only two attitudes: $[V_t^{ii}, V_t^{io}]$, meaning reflexive attitude and attitude towards the others, so that i agents do not discriminate for the other's ethnic group. Suppose that agents act with ethnocentrism so that only the dynamics of the $V_t^{io}, \forall i$ is interesting. As before, in oblique socialization process, each agent is influenced by agents around, and consider with higher weight closer ethnic groups following the cultural distance rule.
Since now the problem is unidimensional, order all the attitudes in increasing order such that the folllowing vector arises $V_t = [V_t^1, V_t^2, ..., V_t^{i-1}, V_t^i, V_t^{i+1}, ..., V_t^n]$.
Call now $I_t = \{i : V_t^i - V_t^{i-1} > \bar{\Delta}\}$. This is the set of all the V_t^i that does not have a link with the left-neighbour. This means that V_t^{i-1} belongs to a set of attitudes whose dynamics is indepenndent from the dynamics of V_t^i. As a consequence, if we indicate the cardinality of the set as $|I_t|$, we have that the number of groups that are independent in their dynamics at each time is $|I_t^i| + 1$. Thus call fragmentation of the society at time t $\Phi_t = |I_t^i| + 1$.

Proposition 8. *Under Basic Cultural Distance rule, fragmentation can never decrease during time.*

Proof. See Appendix B. ☐

Last proposition basically states that $\lim_{k \to \infty} |I_{t+k}| \geq |I_t|, \forall t$. This is an important result since we prove that this kind of homophily rule may be an obstacle towards integration of values. In particular we could be interested in some conditions under which integration may occur in the long run and when, on the contrary, this cannot happen. The followng corollary hepls us:

Corollary 10: *A necessary condition for integration to happen in the long run is that* $\Phi_0 = 1$ *or, equivalently,* $I_0 = \emptyset$.

This necessary condition is very restrictive since it states that, if groups share the same openness parameters, then if there is positive fragmentation at time zero it is impossible to observe integration. This means that only open societies or societies in which each group is close enough to its neighbour there is hope for integration. It has to be noticed that it is not needed that every group is close to every other. Thus the corollary imposes some conditions on the distribution of initial values. These results are driven by the fact that two neighbour ethnic groups that are too distant in order to be influenced each other, can never become closer. Thus two groups can never merge together, but one group can be divided into two groups during time. It is then interesting to find some condition for this to happen. The following corollary states in fact:

Corollary 11: *A necessary condition for* $\Phi_t < \Phi_{t+1}$ *in convergence is that* $\exists i : V_t^{i+1} - V_t^i < \bar{V}_t^{i+1} - \bar{V}_t^i$.

The first part of the condition states that if V_t^i and V_t^{i+1} belong to the same influence group, then V_t^i should me more influence by its left neighbors, while V_t^{i+1} must be more influenced by its right-neighbors. This may happen if V_t^i and V_t^{i+1} are the ending points of two different clusters. Said differently, if a group is polarized in two clusters, then this may drive to the creatiion of two different groups. We should note that this however may also not end up in the division into two groups, and thus leading simpli to a non monotonicity in convergence of attitudes.This necessary condition may also be satisfied if V_t^i and V_t^{i+1} belong to the same cluster but the second on is much more influenced by its right neighbors that the first one.

We now turn the analysis to the more complex case in which agent can discriminate in their atitudes towards all the groups and not only between themselves and the others.

We can state the following propositions:

Proposition 9. *If $\bar{\Delta} = 1$ or $\bar{\Delta} = 0$, given Basic Cultural Distance then any GCSF Dynamics converges.*

Proof. See Appendix B. □

Proposition 10. *If there are only two ethnic groups, then given Basic Cultural Distance then convergence happens.*

Proof. See Appendix B. □

We cannot provide a mathematical proof for convergence a generic number ($n > 2$) of ethnic groups and a generic $\bar{\Delta}$. However we have run a big number of simulations with different intial values of any parameters and convergence always happened. Thus, a deeper mathematical analysis on this aspect is needed.
Last propositions and simulations state that if every couple of groups share the judgment over reciprocal similarity, so that they both feel dissimilar or similar each other, whatever the degree of this similarity, then convergence happens. In particular, if $\bar{\Delta}$ is low, a hierarchy equilibrium is likely to be observed. Take, in fact, matrix B: every block consists of all values of all ethnic groups linked by a certain degree of similarity. In particular if $s^{ik} = 0$ but $s^{ij} > 0$ and $s_{jk} > 0$ then the values of these three ethnic groups produce an irreducible block thus being a diagonal block in the B matrix. Then, as in the other hierarchy equilibria, every block converge to a different value, so that ethnic groups inside each block integrate among them. This means that, while in the previously studied case ethnic social ranking and reciprocity could not be shown together, this may happen here. In fact if $w_{t,ij}^{ji} > 0$ then V^{ij} and V^{ji} belong to the same diagonal block, thus they may converge to the same value. In the meanwhile different blocks converge to different long run values so that a ranking may rise. Moreover last propositions also states that it is not given that, if two groups are at time t on an integration pattern then they will integrate. Consider the case in which an irreducible submatrix is composed by two clusters of elements. In each cluster similarity among groups is high, but only few elements in each cluster are connected with elements of the other cluster with very low similarity. Then it can be that at the first steps each cluster would temporarily moves towards its mean. In this way there is the possiblity of breaking the links between the clusters so that two independent submatrices may rise. In this way every cluster will then converge to a different level.
Last propositions thus makes us able to understand why minorities may follow dfferent integration patterns. Empirical evidence shows that european immigrants in US got integrated faster than other minorities. In this way we explain that this may had happened since values of european minorities were much closer to US ones than other ethnic groups. This had brought to higher similarity perception between european and US people, and thus to narrower linkages towards integration.
These propositions also clarifies a new role for vertical socialization. Differently from the cases in which weights were exogenous, in this case there is no need of positive vertical socialization in order to get convergence. Vertical socialization had the role to make vales at $t + 1$ rooted at their counterpart at time t, so to avoid cyclic matrices. If, then, paramenters are endogenous, since self-similarity is always positive, then $V_{t+1}^{ij} = f(V_t^{ij})$ always happens. In fact, children may find own parents' values during oblique socialization as well, so that vertical socialization is not the only place in which this may happen. Thus, even if parents decide to produce a suboptimal effort level, convergence happens as well. For sure final level will be different. However we have that τ_t^i and $w_{t,ij}^{ij}$ are substitutes since, if $w_{t,ij}^{ij}$

is high, parents may decide to give up part of their effort since the same values may be tken by the children from the other forms of socialization.
Let's now analyse the effect of the Openness Parameter $\bar{\Delta}$: if it is very low we are in front of what we can identify as an exclusive similarity, meaning that agents are very demanding in terms of value similarity in order to consider others in their attitude formation scheme. An inclusive similarity, on the other hand, holds if the threshold is high so that agents are not so demanding in terms of similarity in order to question their own values and consider other's attitudes in their socialization process.
In order to better understand the role if this parameter consider the matrices below that report an example. The first one represents the starting values, while the other three represent the equilibrium values for different levels of $\bar{s} = 1 - \bar{\Delta}$.

$t=0$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s}=0.9$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s}=0.8$	i	j	k
i	0.63	0.63	0.2
j	0.63	0.63	0.2
k	0.8	0.4	1

$\bar{s}=0.4$	i	j	k
i	0.43	0.43	0.43
j	0.43	0.43	0.43
k	0.43	0.43	0.43

The simulations had been run for $p^i = 0.7$ and $p^j = p^k = 0.15$ so that the difference in the evolution of same size minorities become clearer. Moreover $s_t^{ij} = [1 - \frac{1}{n}\sum_k |V_t^{ik} - V_t^{jk}|]$, with n the number of groups, and $w_t^{ij} = \frac{p_t^j s_t^{ij}}{\sum_k p_t^k s_t^{ik}}$[9].
The original situation is such that there is a bad attitude of i and j agents towards k and $s_0^{ij} = 0.8$, $s_0^{ik} = 0.6$ and $s_0^{kj} = 0.53$. In particular, there are two minorities, one of which (k) is considered bad by te other two groups, while both minorities have a good attitude towards the majority. Now, if $\bar{s}$ is high, agents are very conservatives meaning that they need a high degree of similarity in order to be influenced by others in their attitudes: this is what we call *exclusive similarity.* As a result no change is shown in the long run. This outcome can be considered similar to what it is usually called *'closed society'.* In particular we observe that in this case contacts among agents of different groups are not useful in order to get a higher degree of integration. Thus, it is not enough to make two groups in touch in order to achieve at least a higher integration, if they cannot consider the other group's values in their own values formation process. If $\bar{s}$ is higher then groups begin to be influenced, and, as a result, some groups will share the same attitudes set (i and j in this case), while others (as k) do not change their attitudes. Only for low levels of $\bar{s}$ we have generalized cross influence: this is what we call *inclusive similarity.* In particular an *open society* can be considered a one in which agents are prone towards diversity such that they consider even distant groups in their attitude formation scheme. This open society is the most likely to converge to integration outcomes. Thus interesting links between these similarity thresholds effects and the 'open society' of Popper can be analysed since, if thresholds are low, then the intergroup contamination and the questioning of parents' values is very likely to be similar to the critical thinking and tradition challenges Popper talked about.
The case shown above also helps in the understanding of why even same size minorities may have very different integration patterns. In this case, both minorities have good attitude towards majority, but one of them (i) share with the majority the bad attitude towards the other. Then, if the society is not sufficiently opened, we observe the first two integrating and the misconsidered minority being out of any integration pattern. This numerical example makes us clear why cultural similar groups (i and j) may converge, thus having some insights on the fact that european minorities in the US integrate faster that other non-european groups. In fact if we suppose that European immigrant groups were much more similar to Wasp majority than Black, Asians or Hispanics (and this is reasonable since Wasp are derived from part of the european culture), we can reasonably understand this phenomenon. We should also add that this similarity measure is not entirely endogenous, so that it may take into account some other similarity measures. We can think that some aspect of culture, as religious beliefs, may play a role. Moreover, if we think at

[9] In these simulations we weighted also for the population size. However, since population shares re fixed, then they work only as scalars.

some peculiar historical aspects, as black slavery, this will play for sure a role in the patterns of different ethnic groups. We can thus think that black slavery had an impact on the initial values of the V. Thus, noting that this parameter is such a sensible element in the model, some extensions on how this may change, how it is influenced by institutions and how it can be part of a policy for integration becomes crucially important.

7.2 Group Dependent $\bar{\Delta}$

In this framework another interesting insight is the possibility that the openness parameter is group dependent so taking values $\bar{\Delta}^i$. This means that two ethnic groups may differ in their openess degree. Even though this threshold is considered here as exogenous, it can be a function of the socio-economic role of the group in the society. If the group is loosing position in the economic position scale, for example, it could be reasonable to observe lower levels of openess since a fear of loosing values may arise. On the other hand, if a group is experiencing a gain in socio-economic positions then agents may become more open for the opposite reason. Thus, depending on the thresholds levels and the initial similarity values, it could happen that $s^{ij} \neq s^{ji}$ so that non symmetric socialization rules holds: this happens if the similarity is above the threshold for one group and below for the other[10].
As in the previous case, we first restrict the analysis to te case in which agents may only discriminate, in their attitudes, between them and the others. We should re-definte the set I_t since now the similarity is no more symmetric and thus the comonents of the network are not naturally strongly connected. We thus define: $I_t = \{i : V_t^i - V_t^{i-1} > \bar{\Delta}^i \wedge V_t^i - V_t^{i-1} > \bar{\Delta}^{i-1}\}$. In this way the same reasoning as before holds here. Thus, as before, we can state that:

Corollary 12: *Under Basic Cultural Distance rule and group dependent openness parameter, fragmentation can never decrease during time. A necessary condition for integration to happen in the long run is that* $\Phi_0 = 1$ *or, equivalently,* $I_0 = \emptyset$. *A necessary condition for* $\Phi_t < \Phi_{t+1}$ *in convergence is that* $\exists i : V_t^{i+1} - V_t^i < \bar{V}_t^{i+1} - \bar{V}_t^i$.

Thus, as it is evident, the presence of the homogenous or heterogeneous openness parameters does not infuence the fragmentation properties of the model. Still this heterogeneity can capture some interesting phenomena. What we can observe in interaction is that, depending on how the dynamics proceedes, some agent that do not consider others attitude at the beginnnig, if they become closer, may consider them, or the opposite may happen. Thus:

Definition 4: *Define a Break for agent* i *the situation in which* $s_t(ij) = 0, \forall j \neq i$ *and* $s_{t+1}^{ij} > 0$ *for at least one* j.

Then we have:

Corollary 13: *A necessary condition for a break for agent* i *is that* $s_t^{ij} = 0, \forall j$,

- *and either* $\exists k > 0$ *such that* $\sum_n s_t(i+k, i+1-n) > \sum_m s_t(i+k, i+k+m)$,
- *or* $\exists k < 0$ *such that* $\sum_n s_t(i+k, i+k-n) < \sum_m s_t(i+k, i-1+m)$.

This corollary means that, in order to observe a break for i there must exist another point assigning an higher similarity to i and its neighbors such that its oblique socialization lies between $i-1$ and $i+1$. Only in this way it is possible that the similarity between i and k may become positive sine k has the chance to become closer to k than its previous closest neighbors.

We now turn the multidimensional case in order to complete the analysis. Consider again the previous initial situation in which there is a majority i, and two minorities such that one of them, j, is similar to the majority, and the other

[10] This same fact may happen if similarity function is group specific, but we do not consider this case here.

one, k, that is less similar, while the degree of similarity between the minorities is very low. We analyse now the cases in which one of these groups, in turn, shows a high level of openess ($\bar{s} = 0.4$), while the other two shows a high level of closeness ($\bar{s} = 0.8$). The graphs below show the cases in which in turns, i, j, and k respectively have a low opennes parameter level, while the tables below show the equilibrium values.
The first case represents the one in which the majority is open: we can identify two periods in the convergence. In the first one we have that one ethnic group does not feel similar to anyother and thus it has no contamination nor dynamics: this is the k group for the first 20 generations. After this period of time i agents (and j agents through i's influence) became closer such that now both j and k begin to include the others in the socialization scheme experiencing the convergence of the second period. This irregularity in convergence makes again clearer the role of this parameter in the understanding of short run cross-influences: having in fact a short run view over the dynamics it could be thought that k agents would never wanted to integrate in the society. It was then enough, in this case, to have one group (i) that uses inclusive similarity and is felt similar to a group using exclusive similarity (j) in order to create a bridge for long run integration. In this case, i agents have to be patient and waint for almost 20 generations before hving the first results of their openness: this gives the idea that integration processes may not be a matter of yearsbut o decades or more.

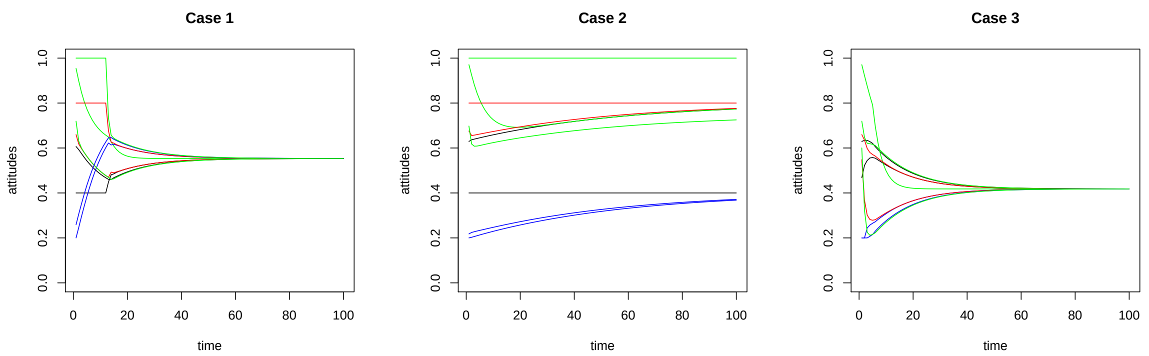

Figure 6: Simulations with endogenous weights

$t = 0$	i	j	k
i	1	0.6	0.2
j	0.8	1	0.2
k	0.8	0.4	1

$\bar{s}^i = 0.4$	i	j	k
i	0.55	0.55	0.55
j	0.55	0.55	0.55
k	0.55	0.55	0.55

$\bar{s}^j = 0.4$	i	j	k
i	0.8	0.8	0.4
j	0.8	0.748	0.4
k	0.8	0.4	1

$\bar{s}^k = 0.4$	i	j	k
i	0.42	0.42	0.42
j	0.42	0.42	0.42
k	0.42	0.42	0.42

The second case represents the situation in which the minority closer to the majority is open: then even though some influences from the k groups happen to be observed, in the long run i and j integrate almost perfectly, while the other minority group remains segregated. This also happens for a total open minority: in fact, since j agents are more similar to i than to k, then they will always take more care of i values that of k values, so that their mean will always be biased in favour of the firsts. Consequently it never happens that i and k become sufficiently close to be influenced each other. If, on the other hand, the most dissimilar minority is opened (as in the third case), then integration happens since the ethnicity that was an obstacle for integration removes the closeness prejudice.
Even if then we do not provide graphs for the dynamics of the optimal socialization effort, in these cases it happens that non monotonicity happens. This is due to the fact that any time a link is built or it disappear, then the oblique socialization mechanism changes and thus new values enter in the socialization function with effect on the parents' loss. If then we combine this endogeneity with population dynamics more sources of non monotonicity may be observed.
These three simple numerical examples are just indicative of three phenomena that can happen during the integration process. We can thus capture the fact that we can observe strong changes in the integration process with groups that

do not integrate until a given similarity level is reached, and other groups working as bridge-builders across ethncities. We can give reason of the fact that the most dissimilar group remains segregated even in the case in which other groups are opened towards it, since tighter links hold among most similar groups.
As in the previous case, we run a great number of simulation with different values for the all the parameters involved in the dynamics and we have always observed convergence. Even though we are not able to provide a mathematical proof for this convergence, looking at the simulation that cover a wide range of possibilities, we are confident that convergence happens also for group specific openness levels. With this last part we make clear the effect that heteroegenous propensities towards openness may have on the final outcome. We are consciuos that these measures may be endogenous, but for the time being we consider them as dependent on something out of the model, and dependent on some socio-economic position of the group, as previously argued. However these last results make clear the role that a policy focused on making people more open and tolerant may have on integration policies, since it comes clear that, besides material factors, these are crucial elements of the problem. Moreover, the endogenization of the socialization rule does not help in explaining why cycles may be observed, but drives us towards the direction of finding them into the changes of the socio-economic position of the groups, thus providing an exogenous explanation for these phenomena. On the other side, racism per se may be a results of the endogenous dynamics, if agents do no show a sufficiently high level of tolerance and openness. If weights are endogenized in order to depend both on similarity and population shares, and population dynamics shows cycles, then it can be that cycles in racism may be consequently observed.

8 Conclusion

Existing economic cultural evolution literature referring to Cavalli-Sforza and Boyd-Richerson studies, mainly focuses on what happens if time invariant cultural values are transmitted from one generation to the other (as in the contributions of Bisin and Verdier) and studies the evolution of population shares under this assumption. Only recently, some interest has been devoted to the study of convergence of non-fixed cultural values and to the introduction of complexity in the vertical socialization processes. The more recent contributions studied the conditions under which a *melting pot* equilibrium happens in terms of long run equilibrium, finding that it may happen if there is a general cross influence among cultural values. They focus on a single cultural value and they study what happens in the long run
Here, starting from the initial intuition of Boyd and Richerson (1985) about the importance of cultural transmission structures, we study what happens if attention is given to the different oblique socialization schemes. Using a framework in which there are ethnic groups and parents trying to transmit their attitudes towards the different groups, we are able to understand what happens if different interaction schemes among ethnicities are considered. Using schemes as Reciprocity and Ethnocentrism we prove that, if all agents use the same socialization scheme, then the society may converge both to integration both to a social hierarchy based on ethnicity, thus deriving equilibria consistent with empirical studies. Turning then to the welfare analysis we found that, in a two groups framework, the minority group always put more socialization effort, changes more in values and thus shows more frustration than the majority group. Moreover we find that large minorities or minorities that do not differ too much in values from the majority are more weak that strict or culturally different minorties. We then analyse what happens if different ethnic groups use different socialization schemes. Using a network-derived framework we underline the role that different groups may have in the convergence process: a groups may play as cultural bridge, cultural hubs and if all groups have equal role, cultural cycles may arise. This framework thus gives an instrument in order to analyse why different minorities may end up with different long run integration equilibria. We do not provide yet an endogenous explanation of the different socialization schemes. Still we can understand what happens under different intergroup relation structures. We then provide the first steps for the endogeneization of socialization structures with some sufficient conditions a weights dynamics may satisfy in order to reach integration or segregation equilibria. Finally we provide conditions for convergence with an endogeous homophily rule.
This study opens new roads in which the reasearch may be run: there is space in order to understand what happens if the structure of the interethnic relationships change with time with different mechanism than what we did here, so providing a new endogeneization of socialization schemes. Similarly it would be interesting to study what happens

if forms of socialization schemes other than reciprocity and ethnocentrism may be implemented. Again it could be interesting to analyse what happens if horizontal socializiation is taken into account into these schemes. An empirical analysis on some case studies, however, may be necessary.

Appendix A: Weights Matrices for Simulations

We report below the weight matrices we used in the simulations for figure 1. With respect to the cases reported in the definition of socialization rules we impose that $w^{jj}_{t,ij} = 0, \forall i, j$ meaning that in forming ij attitude, i agents do not consider the reflexive attitude of j. This does not change the way in which dynamics happens, but just levels. In particular it avoids that in HEE and IEE everything converges to $V = E = 1$, as it is clear from proposition 4-5. Moreover we just write **X** where there is a positive weight. The weights are represented by the population shares so that, for example, $w^{kj}_{ij} = p^k$. Since here population shares are constant, then the weight matrix is fixed.

HE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	**X**	0	0	**X**	0	0
ij	0	**X**	0	0	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	0	0	0
ji	0	0	0	**X**	0	0	**X**	0	0
jj	0	**X**	0	0	**X**	0	0	**X**	0
jk	0	0	**X**	0	0	**X**	0	0	0
ki	0	0	0	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	0	0	**X**	0
kk	0	0	**X**	0	0	**X**	0	0	**X**

IE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	**X**	0	0	**X**	0	0
ij	0	**X**	0	**X**	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	**X**	0	0
ji	0	**X**	0	**X**	0	0	**X**	0	0
jj	0	**X**	0	0	**X**	0	0	**X**	0
jk	0	0	**X**	0	0	**X**	0	**X**	0
ki	0	0	**X**	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	**X**	0	**X**	0
kk	0	0	**X**	0	0	**X**	0	0	**X**

HEE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	0	0	0	0	0	0
ij	0	**X**	0	0	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	0	0	0
ji	0	0	0	**X**	0	0	**X**	0	0
jj	0	0	0	0	**X**	0	0	0	0
jk	0	0	**X**	0	0	**X**	0	0	0
ki	0	0	0	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	0	0	**X**	0
kk	0	0	0	0	0	0	0	0	**X**

IEE	ii	ij	ik	ji	jj	jk	ki	kj	kk
ii	**X**	0	0	0	0	0	0	0	0
ij	0	**X**	0	**X**	0	0	0	**X**	0
ik	0	0	**X**	0	0	**X**	**X**	0	0
ji	0	**X**	0	**X**	0	0	**X**	0	0
jj	0	0	0	0	**X**	0	0	0	0
jk	0	0	**X**	0	0	**X**	0	**X**	0
ki	0	0	**X**	**X**	0	0	**X**	0	0
kj	0	**X**	0	0	0	**X**	0	**X**	0
kk	0	0	0	0	0	0	0	0	**X**

Appendix B: Proofs of the Propositions

Proof of Proposition 1

Proof. Since $\tau^i_t \in [0,1]$ then $V^{ij}_{t+1}(\tau^i, \bar{p}_t, \bar{V}_t)$ is continous in τ so that W^{ii}_t is continuous in τ^i_t. Since $c(\tau^i_t)$ is also continuous in τ^i_t then $W^i_{t,i} - c(\tau^i_t)$ admits a global maximum in $\tau^i_t \in [0,1]$ so that $\tau^{i^*}_t$ exists. Moreover $W^i_{t,i}$ can be written as $V^* - (1 - \tau^i_t)^2 \sum_k (V^{ik}_t - \bar{V}^*_t)^2$ and is strictly concave in τ^i_t. $c(\tau^i_t)$ is strictly convex in τ^i_t then $W^i_{t,i} - c(\tau^i_t)$ is strictly concave in τ^i_t and so $\tau^{i^*}_t$ is unique.
Suppose now that $V^{ij}_t = \bar{V}^{ij}_t, \forall j$, then $\frac{\partial W^i_t}{\partial \tau^i_t} = 0, \forall \tau^i_t$.
In order to be $\tau^{i*}_t = 1$ it should be $c'(1) \leq \frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=1}$. But $c'(1) > 0$, while $\frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=1} = 0$ so that this is impossible. Moreover it cannot be $\tau^{i*}_t \in (0,1)$ since at the optimum it should be $\frac{\partial W^i_t}{\partial \tau^i_t} = c'(\tau^i_t)$, but $\frac{\partial W^i_t}{\partial \tau^i_t} = 0, \forall \tau^i_t$ and $c'(\tau^i_t) > 0, \forall \tau^i_t > 0$.
Thus $\tau^{i*}_t = 0$ since $c'(0) = \frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=0}$ so that $c'(0) \geq \frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=0}$.
Suppose now that $V^{ij}_t \neq \bar{V}^{ij}_t$ for at least one j. In this case $\frac{\partial W^i_t}{\partial \tau^i_t} > 0 \forall \tau^i_t < 1$, and $\frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=1} = 0$.
In order to be $\tau^{i*}_t = 0$ it should be $c'(0) \geq \frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=0}$. But $c'(0) = 0$, while $\frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=0} > 0$ so that it is impossible.
In order to be $\tau^{i*}_t = 1$ it should be $c'(1) \leq \frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=1}$. But $c'(1) > 0$, while $\frac{\partial W^i_t}{\partial \tau^i_t}|_{\tau^i_t=1} = 0$ so that it is impossible. Thus it must be $\tau^{i*}_t \in (0,1)$.

□

Proof of Proposition 2

Proof. Equation (6) can be written as
$V^{ij}_{t+1} = (\tau^i_t + w^{ij}_{ij} - w^{ij}_{ij}\tau^i_t)V^{ij}_t + (1-\tau^i_t)(\sum_{k,w} w^{kw}_{ij}V^{kw}_t)$. Since, by proposition 1, $\tau^{i^*}_t(\bar{p}_t, \bar{V}_t) \in (0,1)$ and is endogenous, then the dynamics is not linear in $\bar{V}_t$. Consider now any $\tau^i_t \in (0,1)$ exogenously given at any time and for each group such that $\tau^{i^*}_t(\bar{p}_t, \bar{V}_t) \in (0,1)$ is only one possible value for τ^i_t. We will prove convergence for any τ^i_t such that convergence for $\tau^{i^*}_t(\bar{p}_t, \bar{V}_t)$ is only a specific case and thus convergence will also happen for every suboptimal $\tau^i_t \in (0,1)$.
Order the type entries in order to get a (n^2X1) vector as the following:

$$\bar{V}_t = [V^{ii}_t, V^{ij}_t, ..., V^{in}_t, V^{ji}_t, V^{jj}_t, ..., V^{jn}_t, ..., ..., V^{ni}_t, ..., V^{nn}_t].$$

We can thus write the dynamics as

$$V^{ij}_{t+1} = [a^{ii}_{t,ij}, a^{ij}_{t,ij}, a^{ik}_{t,ij}, ..., a^{nn}_{t,ij}]\bar{V}^{'}_t$$

in which the non-diagonal terms $a^{kw}_{t,ij} = (1-\tau^i_t)w^{kw}_{ij}$ if $i \neq k$ and the diagonal term $\forall j \neq w$, and $a^{ij}_{t,ij} = (\tau^i_t + w^{ij}_{ij} - w^{ij}_{ij}\tau^i_t), \forall i, j$. We thus have the following linear system: $V^{'}_{t+1} = AV^{'}_t$, in which A_t is the (n^2Xn^2) matrix in which the entries are the $a^{kw}_{t,ij}$, $\forall i, j, k, w$, so that A is row-normalized.
Consider now the A matrix. Since $\tau^i_t > 0$, then $a^{ii}_t > 0 \ \forall i$ $(*)$.
Brueckner and Smirnov (2007, 2008) proved that given a linear system $V'_{t+1} = AV'_t$, then if A is irreducible[11] for all t, if at least one diagonal element is positive, then the matrix is also acyclic[12], and thus the dynamics converges to a steady state. Since weights w^{kw}_{ij} are time independent, and $\tau^i_t \in (0,1)$, the matrix A is always irreducible or always not reducible. Moreover, since $(*)$ holds the matrix, if irreducible, is acyclic. Consequently Brueckner and Smirnov (2007, 2008) immediately applies when the matrix A is irreducible. Moreoever, in this case, they prove that all elements converge to the same value. If A is not irreducible, it can always be rewritten as an upper-triangular-block matrix as B by means of columns and rows transpositions:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 0 & ... & ... & ... \\ 0 & 0 & 0 & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & 0 & b_{nn} \end{bmatrix}$$

in which every b_{ii} is a square block, while some non-diagonal blocks may have all zero entries.
Given time independent weights and $\tau^i_t \in (0,1)$, the strucure of the B matrix is time invariant $\forall t \geq T$. If B is a block-diagonal matrix, and thus all non-diagonal blocks have all zero entries, then, since every diagonal block is irreducible and, by (*), acyclic, then every element of each block converges and thus overall convergence happens.
If B is not a block diagonal matrix, take the b_{nn} block. Again the structure of the matrix is time invariant. b_{nn} elements thus converge since it is irreducible and acyclic.
Consider now the $b_{n-1,n-1}$ block, and analyse its dynamics that depends only on $b_{n-1,n-1}$ and b_{nn} blocks' elements. Consider first the case in which the b_{nn} block is composed of only one element. Then consider the submatrix for the last two blocks:

$$\begin{bmatrix} b_{n-1,n-1} & b_{n-1,n} \\ 0 & b_{nn} \end{bmatrix}$$

Then the corresponding weight matrix is represented as follows:

[11] A square matrix A is irreducible if and only if for each i and j there exists some k such that $(a_{ij})^k > 0$, with $(a_{ij})^k$ being the ij entry of the k^{th} power matrix of A. Moreover a matrix is irreducible if and only if the digraph associated to A is strongly connected.

[12] Call d_{ii} the period of the a_{ii} element of the A square matrix. d_{ii} is the greatest common divisors among all k such that $(a_{ij})^k > 0$. A square matrix A is acyclic if and only if $d_{ii} = d_{jj} = 1$, $\forall i, j$.

$$B_t = \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 1-\sum_i \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 1-\sum_i \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 1-\sum_i \gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $X_{t+1} = B_t X_t = B^t X_0$. In terms of Markov processes, this can be identified as a *Non-Homogenous single-unireducible Markov Process*. Given the structure of the process, the limit probability of the markov process represented by the transmission matrix is also the limit of the matrix of weights. Consequently if the limit probability of the markov process exist, then the process converges, and if the limit probability can be identified, then the limit of the matrix of weights can be identified too. D'amico et al. (2009) proved that, if the probability matrix is non-homogenous sigle-unireducible, then

$$\lim_{t\to\infty} B^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that convergence to the b_{nn} element level happens.

Consider now the case in which the b_{nn} block is composed of more that one element. Since the b_{nn} block is strongly connected and there are no influences by entries not belonging to the block, then all elements of the b_{nn} block converge to the same value I, as proved by Brueckner and Smirnov (2007, 2008). Consequently we have that (for the case of a two-element block, but it can be extended to a n-element case):

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

Thus, we can rewrite the limit of the dynamics of the first $n-1$ entries as:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t}+\alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t}+\beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t}+\gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \end{bmatrix}$$

This is again the limit of a non-homogenous single-unireducible markov process, so that we have:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}.$$

If in the whole B matrix the b_{nn} block is the only block that does not depend on any other block, then all other block directly or indirectly depend only on it, so that the same proof can be used to prove convergence of all the entries of the B matrix to the convergence level of the b_{nn} block. Suppose now that this is not the case, and so that $b_{n-1,n}$ block has all zero elements and thus both b_{nn} and $b_{n-1,n-1}$ blocks have dynamics independent each other.

Consider now the $b_{n-2,n-2}$ block. If $b_{n-2,n-1}$ block or $b_{n-2,n}$ block has all zero entries, then the previous result applies for the convergence of $b_{n-2,n-2}$ elements too, since all $b_{n-2,n-2}$ elements have a dynamics that depends on the elements of own block and on a converging block. If this is not the case, then take any V^{ij} belonging to the $b_{n-2,n-2}$ process. Then, we can always find weights α_t, β_t and γ_t such that any element V_t^{ij} of this process can have its dynamic rewritten as

$$V_t^{ij} = \alpha_t V_t^{ij} + \beta_t \bar{b}_{n-2,n-2} + \gamma_t \bar{b}_{n-1,n-1} + \delta \bar{b}_{nn} \qquad (7)$$

with $\alpha_t \in (0,1]$ (since $\tau_t^i > 0$), $\beta_t \in \Re^+$, $\gamma_t \in \Re^+$ and $\delta_t \in \Re^+$ in which $\bar{b}_{n-2,n-2}$ is the value at which the elements of the $b_{n-2,n-2}$ diagonal block had converged if the dynamics would have been given only by this diagonal block, $\bar{b}_{n-1,n-1}$ is the value at which the elements of the $b_{n-1,n-1}$ diagonal block had converged if the dynamics would have been given only by this diagonal block and $\bar{b}_{nn}$ is the convergence points of the block b_{nn}.[13]

Now, for entries for which $w_{t,ij}^{ij} = 1$ then $\alpha_t = 1, \beta_t = \gamma_t, \delta_t = 0$. Since weights are fixed, then this holds for all periods, so that these entries do not show any dynamics.

Consider now entries with $w_{t,ij}^{ij} \neq 1, \forall i, j$. In this case we can find $\alpha_t \in (0,1)$, $\beta_t \in \Re^+$, $\gamma_t \in \Re^+$ and $\delta_t \in \Re^+$.

Define $\alpha_t! \equiv \prod_{i=0}^{t} \alpha_i$. We can write:

$$\begin{aligned}
V_1^{ij} &= \alpha_1 V_0^{ij} + \beta_1 \bar{b}_{n-2,n-2} + \gamma_1 \bar{b}_{n-1,n-1} + \delta_1 b_{nn} \\
V_2^{ij} &= \alpha_2\alpha_1 V_0^{ij} + \alpha_2\beta_1 \bar{b}_{n-1,n-1} + \alpha_2\gamma_1 \bar{b}_{n-1,n-1} + \alpha_2\delta_1 \bar{b}_{nn} + \beta_2 \bar{b}_{n-2,n-2} + \gamma_2 \bar{b}_{n-1,n-1} + \delta_2 \bar{b}_{nn} \\
V_3^{ij} &= \alpha_3\alpha_2\alpha_1 V_0^{ij} + \alpha_3\alpha_2\beta_1 \bar{b}_{n-1,n-1} + \alpha_3\alpha_2\gamma_1 \bar{b}_{n-1,n-1} + \alpha_3\alpha_2\delta_1 \bar{b}_{nn} + \alpha_3\beta_2 \bar{b}_{n-2,n-2} \\
&\quad + \alpha_3\gamma_2 \bar{b}_{n-1,n-1} + \alpha_3\delta_2 \bar{b}_{nn} + \beta_3 \bar{b}_{n-2,n-2} + \gamma_3 \bar{b}_{n-1,n-1} + \delta_3 \bar{b}_{nn} \\
... &= ... \\
V_t^{ij} &= V_0^{ij} \prod_{i=1}^{t} \alpha_i + \bar{b}_{n-2,n-2}(\beta_t + \beta_{t-1}\alpha_t + \beta_{t-2}\alpha_{t-1}\alpha_t + \beta_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \beta_1\alpha_t\alpha_{t-1}...\alpha_t) \\
\Downarrow &\quad + \bar{b}_{n-1,n-1}(\gamma_t + \gamma_{t-1}\alpha_t + \gamma_{t-2}\alpha_{t-1}\alpha_t + \gamma_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \gamma_1\alpha_t\alpha_{t-1}...\alpha_2) \\
\Downarrow &\quad + \bar{b}_{n,n}(\delta_t + \delta_{t-1}\alpha_t + \delta_{t-2}\alpha_{t-1}\alpha_t + \delta_{t-3}\alpha_{t-2}\alpha_{t-1}\alpha_t + ... + \delta_1\alpha_t\alpha_{t-1}...\alpha_2) \\
V_t^{ij} &= V_0^{ij} \prod_{i=1}^{t} \alpha_i + \bar{b}_{n-2,n-2} \sum_{i=1}^{t} \beta_i \frac{\alpha_t!}{\alpha_i!} + \bar{b}_{n-1,n-1} \sum_{i=1}^{t} \gamma_i \frac{\alpha_t!}{\alpha_i!} + \bar{b}_{nn} \sum_{i=1}^{t} \delta_i \frac{\alpha_t!}{\alpha_i!}
\end{aligned}$$

In order to analyse the convergence we should look at $\lim_{t\to\infty} V_t^{ij}$.

Trivially $\lim_{t\to\infty} V_0^{ij} \prod_{i=0}^{t} \alpha_{i+1} = 0$ since $\alpha_t \in (0,1)$.

Consider now the term $\bar{b}_{n-2,n-2} \sum_{i=1}^{t} \beta_i \frac{\alpha_t!}{\alpha_i!}$ and call $z_j = \beta_j \frac{\alpha_t!}{\alpha_j!}$.

$j = t \Rightarrow z_j = \beta_t$
$j = t-1 \Rightarrow z_j = \beta_{t-1}\alpha_t$
$j = t-2 \Rightarrow z_j = \beta_{t-2}\alpha_t\alpha_{t-1}$
$j = t-3 \Rightarrow z_j = \beta_{t-3}\alpha_t\alpha_{t-1}\alpha_{t-2}$
...
$j = t-t+1 \Rightarrow z_j = \beta_{t-t+1}\alpha_t\alpha_{t-1}\alpha_{t-2}...\alpha_{t-t+2}$

take $\bar{\beta} = max\{\beta_t\}$ and $\bar{\alpha} = max\{\alpha_t\}$ then

$$\sum_{j=0}^{t} \beta_j \frac{\alpha_t!}{\alpha_j!} \leq \sum_{j=0}^{t} \bar{\beta} \frac{\alpha_t!}{\alpha_j!} = \bar{\beta} \sum_{j=0}^{t} \frac{\alpha_t!}{\alpha_j!} = \bar{\beta}(1 + \alpha_t + \alpha_t\alpha_{t-1} + \alpha_t\alpha_{t-1}\alpha_{t-2} + ...) \leq \bar{\beta}(\sum_{j=0}^{t} \bar{\alpha}^j).$$

Now, $\lim_{t\to\infty} \sum_{j=0}^{t} \bar{\alpha}^j = \frac{1}{1-\bar{\alpha}}$.

Consequently, $\sum_{j=0}^{t} \beta_j \frac{\alpha_t!}{\alpha_j!}$ is an increasing sequence bounded above by a converging sequence, so that it converges.

The same proof holds for $\sum_{j=0}^{t-1} \gamma_{j+1} \frac{\alpha_t!}{\alpha_{j+1}!}$ and for $\sum_{j=0}^{t-1} \delta_{j+1} \frac{\alpha_t!}{\alpha_{j+1}!}$.

[13] Notice that this is not a new dynamic, but is a way in order to find coefficients such that these V_t^{ij} exactly correspond to the values indicated by the original dynamics. In this way the convergence points of the two rules coincide since the second one is built in order to coincide step by step with the original one. Consequently proving convergence for the second one implies proving convergence for the original dynamics.

Thus $\lim_{t\to\infty} V_t^{ij}$ is a weighted finite sum of converging series so that it converges too. Thus the elements of the $b_(n-2, n-2)$ block converge. If we recursively apply this reasoning to all the other blocks until we reach the (b_{11}) block, then convergence to a steady state is proved.

□

Proof of Proposition 3

Proof. A system like $V'_{t+1} = AV'_t$ converges if each diagonal block of the transmission matrix is irreducible, and thus has a strongly connected directed graph, and acyclic. In all cases, socialization rules imply $w_{ij}^{ij} > 0, \forall i, j$, such that all the diagonal blocks of the transmission matrix are acyclic. If Emulation Rule holds, we have that all the entries V^{ij} $\forall i$, with some row and columns transpositions, form a single block which has a strongly connected digraph, since all the links are bidirectional, so that it is also an irreducible block. Thus, by Brueckner and Smirnov (2007, 2008), convergence to a common value for each block, and thus to a HE, happens.
The Reciprocity rule differs from the previous one since each of the previous blocks is connected to the other ones via the double links between V^{ij} and V^{ji} $\forall i, j$, so that all V^{ij} $\forall i, j$ forms a single strongly connected digraph and, for the same reason as before, convergence to a IE happens.
The Ethnocentrism Rule and Reciprocity and Ethnocentrism rule differ from the previous ones in the sense that $V^{ii} = E$ $\forall i$ such that each of these reflexive elements forms a diagonal block per se and do not show any dynamics. The remaining elements have a structure as in the previous two cases thus convergence respectlivey to HEE and IEE happens. □

Proof of Proposition 4

Proof. Given the cost function we have $\tau_t^{i^*} = \frac{(1-p_t^i)^2 (V_t^i - V_t^j)^2}{1+(1-p_t^i)^2 (V_t^i - V_t^j)^2}$. Thus $\tau_t^{i^*} > \tau_t^{j^*}$ if and only if $p_t^i < p_t^j$ and thus if $\tau_t^{i^*} < \frac{1}{2}$. Given the optimal efforts we have that $V_t^i - V_{t+1}^i = \frac{(1-p_t^i)^2 (V_t^i - V_t^j)^2}{(1+(1-p_t^i)^2 (V_t^i - V_t^j)^2)^2}$. Thus $\Delta_t^i = (\frac{(1-p_t^i)^2 (V_t^i - V_t^j)^2}{(1+(1-p_t^i)^2 (V_t^i - V_t^j)^2)^2})^2$. So $\Delta_t^i > \Delta_t^j$ if and only if $p_t^i < p_t^j$ so that if and only if $p_t^i < \frac{1}{2}$. Since then $F_t^i = \Delta_t^i + c(\tau_t^{i^*})$, given that cost functions are increasing in the effort, $F_t^i > F_t^j$ if and only if $p_t^i < \frac{1}{2}$. □

Proof of Proposition 5

Proof. Each component forms a dynamics system per se and thus may be considered separately from the others. Consider first the case in which the elements of a group are strongly connected. Then the transmission matrix may be represented as a block diagonal matrix in which each block is irreducible and thus convergence happens. If there is one sink then it can be represented as a upper triangular block matrix in which the sink is the right-bottom block. From proposition 2 in this case convergence holds too. In order to prove that all the elements of the same component converge to the same value, consider first the case in which the sink is composed by only one node. Then the weight matrix is represented as follows:

$$B_t = \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 1 - \sum_i \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 1 - \sum_i \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 1 - \sum_i \gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that $X_{t+1} = B_t X_t = B^t X_0$. In terms of Markov processes, this can be identified as a *Non-Homogenous single-unireducible Markov Process.* Given the structure of the process, the limit probability of the markov process represented by the transmission matrix is also the limit of the matrix of weights. Consequently if the limit probability of the markov process exist, then the process converges, and if the limit probability can be identified, then the limit of the matrix of weights can be identified too. D'amico et al. (2009) proved that, if the probability matrix is non-homogenous sigle-unireducible, then

$$\lim_{t\to\infty} B^t = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that convergence to the sink level happens.

Consider now the case in which the sink is composed of more that one element. Since the sink is strongly connected and there are no influences by nodes not belonging to the sink, then they converge to the same value I. Consequently we have that (for the case of a two-nodes sink, but it can be extended to a n-nodes sink case):

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & 0 & \alpha_{i-1t}+\alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & 0 & \beta_{i-1t}+\beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & 0 & \gamma_{i-1t}+\gamma_{it} \\ ... & ... & ... & ... & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

Since the last two elements do not show any dynamics and they are fixed on the same values, we can rewrite the limit of the dynamics of the first $n-1$ entries as:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t}+\alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t}+\beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t}+\gamma_{it} \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \end{bmatrix}$$

This is again the limit of a non-homogenous single-unireducible markov process, so that we have:

$$\lim_{t\to\infty} \begin{bmatrix} \alpha_{1t} & \alpha_{2t} & \alpha_{3t} & ... & \alpha_{i-1t} & \alpha_{it} \\ \beta_{1t} & \beta_{2t} & \beta_{3t} & ... & \beta_{i-1t} & \beta it \\ \gamma_{1t} & \gamma_{2t} & \gamma_{3t} & ... & \gamma_{i-1t} & \gamma_{it} \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \delta_{i-1t} & \delta_{it} \\ 0 & 0 & 0 & 0 & \theta_{i-1t} & \theta_{it} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ w_t \\ s_t \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ ... & ... & ... & ... & ... & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ ... \\ I \\ I \end{bmatrix}$$

□

Proof of Proposition 6

Proof. Since all elements forms a single component the proof for convergence is the same as the one for convergence of each component in proposition 4 □

Proof of Proposition 7

Proof. Suppose that ethnocentrism does not hold. If assumptions 3 and 4 hold then the matrix can always be rewritten as a block diagonal matrix in which, at each time, each block is acyclic (since $w^{ij}_{t,ij} > 0, \forall i, j$) and irreducible (because of simmetry). By assumption 4, then, after time T the block structure of the matrix is time invariant. Thus, by Brueckner and Smirnov (2007, 2008), convergence happens. If ethnocentrism hold, then the reflexive traits $V^{ii}, \forall i$ are fixed and each of them form a block with $w^{ii}_{t,ii} = 1$ and thus no dynamics is shown, while the first part of the proof holds for the dynamics of all the other entries.

We prove now the second part of the proposition. If assumption 4 holds, then the matrix structure is stable after time T. Each component of the directed graph forms a dynamics system per se and thus may be considered separately from the others. Consider first the case in which the elements of a component are strongly connected at time T, so that they will always be strongly connected. Then the transmission matrix may be represented as a block diagonal matrix in which each block is irreducible and thus, by Brueckner and Smirnov (2007, 2008) convergence happens. If there is one sink then the matrix can be represented as a upper triangular block matrix in which the sink is the right-bottom block. Consequently the proof follows as the one for proposition 4. □

Proof of Proposition 8

Proof. Suppose that at time t fragmentation $\Phi_t = n$. Thus, there are n disjoined groups of attitudes. In each groups all attitudes move towards their mean. Take now V_t^i and V_t^{i+1} beloging to two different sets. Then $V_t^i > V_{t+1}^i$ and $V_t^{i+1} < V_{t+1}^{i+1}$. Thus the two sets can never be merged together. □

Proof of Proposition 9

Proof. If $\bar{\Delta} = 1$, then $s_t^{ij} > 0, \forall i, j, t$ so that all ethnicities form one single component, the matrix A is thus irreducible and convergence happens.
If $\bar{\Delta} = 0$, then each group forms a component by its own, apart for the case in which two ethnic groups have the very same entries.. In this case, if $V_t^{ij} = V_t^{ji}$ the two groups show no dynamics in type entries. If $V_t^{ij} \neq V_t^{ji}$ then from $t + 1$ they become different in entries so that each of them form a separate component. Thus convergence happens. □

Proof of Proposition 10

Proof. If $\Delta_t^{ij} > \bar{\Delta}$ then they do not influence each others and thus they stay fixed. If $\Delta_t^{ij} < \bar{\Delta}$ they form an irreducible block and they move towards the SS. If they always remain linked, they converge, if they become dissimilar at some point in time the stay fixed for all the rest of the time. Thus convergence happens. □

References

Arias, E. (2001). Change in nuptiality patterns among cuban americans: Evidence of cultural and structural assimilation. *International Migration Review*, 35-2:525–556.

Bar-Tal, D. (1997). Formation and change of ethnic and national stereotypes: An integrative model. *International Journal of Intercultural Relations.*

Berry, J. (2006). Mutual attitudes among immigrants and ethnocultural groups in canada. *International Journal of Intercultural Relations*, 30(6):719–734.

Berry, J. and Kalin, R. (1979). Reciprocity of inter-ethnic attitudes in a multicultural society. *International Journal of Intercultural Relations*, 3:99–112.

Berry, J. and Kalin, R. (1996). Interethnic attitudes in canada: ethnocentrism, consensual hierarchy and reciprocity. *Canadian Journal of Behavioural Science-Revue Canadienne des Sciences du Comportement*, 28:253–261.

Bertrand, M. and Mullainathan, S. (2004). Are emily and greg more employable than lakisha and jamal? a field experiment on labor .... *American Economic Review.*

Bisin, A., Patacchini, E., Verdier, T., and Zenou, Y. (2006). Bend it like beckham: Identity, socialization and assimilation. *CEPR Discussion Paper*, 5662.

Bisin, A., Topa, G., and Verdier, T. (2004). Religious intermarriage and socialization in the united states. *Journal of Political Economy.*

Bisin, A. and Verdier, T. (2000). beyond the melting pot": Cultural transmission, marriage, and the evolution of ethnic and .... *Quarterly Journal of Economics.*

Bisin, A. and Verdier, T. (2001). The economics of cultural transmission and the dynamics of preferences. *Journal of Economic Theory.*

Bobo, L. and Zubrinsky, C. L. (1996). Attitudes on residential integration: perceived status differences, mere in-group preference, or racial prejudice? *Social Forces*, 74:883–909.

Bogardus, E. S. (1926). Social distance in the city. *Proceedings and Publications of the American Sociological Society*, 20:40–46.

Bogardus, E. S. (1959). *Social Distance*. Los Angeles.

Boyd, R. and Richerson, P. J. (1985). *Culture and the Evolutionary Process*. University of Chicago Press.

Boyd, R. and Richerson, P. J. (2005). *The Origin and Evolution of Cultures*. Oxford University Press.

Brueckner, J. and Smirnov, O. (2007). Workings of the melting pot: Social networks and the evolution of population attributes*. *Journal of Regional Science*.

Brueckner, J. and Smirnov, O. (2008). Social networks and the convergence of population attributes: A generalization. *Journal of Regional Science*.

Carlsson, M. and Rooth, D. (2007). Evidence of ethnic discrimination in the swedish labor market using experimental data. *Labour Economics*.

Casadei, T. and Re, L. (2007). *Differenza Razziale, Discriminazione e Razzismo nelle Societa' Multiculturali*. Diabasis.

Cavalli-Sforza, L. and Feldman, M. (1973). Cultural versus biological inheritance: phenotypic transmission from parents to children.(a theory .... *American Journal of Human Genetics*.

Cavalli-Sforza, L. and Feldman, M. (1981). *Cultural Transmission and Evolution*. Princeton University Press, Princeton.

Currarini, S., Jackson, M., and Pin, P. (2009). An economic model of friendship: Homophily, minorities and segregation. *Econometrica*, 77-4:1003–1045.

D'Amico, G., Janssen, J., and Manca, R. (2009). The dynamic behaviour of non-homogenous single-unireducible markov and semi-markov chains. *in Network, Topology and Dynamics - Springer*.

DeMarzo, P., Vayanos, D., and Zwiebel, J. (2003). Persuasion bias, social influence and unidimensional opinions. *The Quarterly Journal of Economics*, August:909–968.

Duckitt, J. (1992). *The Social Psychology of Prejudice*. Praeger Publishers, New York.

Form, W. (2000). Italian protestants: Religion, ethnicity and assimilation. *Journal for the Scientific Study of Religion*, 39-3:307–320.

Goldoni, M. (2007). *Tra sicurezza e discriminazione: il Racial Profiling*.In Differenza Razziale, Discriminazione e Razzismo nelle Societa' Multiculturali. Diabasis.

Golub, B. and Jackson, M. (2007). Naive learning in social networks: convergence, infuence and wisdom of crowds. *FEEM Working Paper Series*, 64.

Golub, B. and Jackson, M. (2009). How homophily affects learning and diffusion in networks. *FEEM Working Paper Series*, 35.

Hagendoorn, L., Drogendijk, R., Tumanov, S., and Hraba, J. (1998). Inter-ethnic preferences and ethnic hierarchies in the former soviet union. *International Journal of Intercultural Relations*.

Hraba, J., Radloff, T., and Gray-Ray, P. (1999). A comparison of black and white social distance. *The Journal of social psychology*.

Jackson, M. (2006). *"The Economics of Social Networks" in* Advances in Economics and Econometrics, Theory and Applications: Ninth World Congress of the Econometric Society. Cambridge University Press.

Lee, M., Sapp, S., and Ray, M. (1996). The reverse social distance scale. *The Journal of social psychology*.

Listhaug, O. and Strabac, Z. (2008). Anti-muslim prejudice in europe: A multilevel analysis of survey data from 30 countries. *Social Science Research*, 37(1):268–286.

Masuoka, N. (2006). Together they become one: Examining the prediction of panhethnic groups consciousness among asian americans and latinos. *Social Science Quarterly*, 87-5:993–1011.

McAdoo (2006). *Black Families*. Sage Publications.

Ogbu, J. U. (1990). Minority education in comparative perspective. *The Journal of Negro Education*, 59-1:345–57.

Parrillo, V. and Donoghue, C. (2005). Updating the bogardus social distance studies: a new national survey. *The Social Science Journal*.

Pichler, M. (2009). Culture formation and endogenous cultural distance. *WP 403, Institute of Mathematical Economics, Bielefeld Universitat*.

Randall, N. H. and Delbridge, S. (2005). Perceptions of social distance in an ethnically fluid community. *Sociological Spectrum*.

Rooth, D. (2009). Automatic associations and discrimination in hiring: Real world evidence. *Labour Economics*.

Schalksoekar, S., Vandevijver, F., and Hoogsteder, M. (2004). Attitudes toward multiculturalism of immigrants and majority members in the netherlands. *International Journal of Intercultural Relations*, 28(6):533–550.

Schellings, T. C. (1971). On the ecology of micromotives. *The Public Interest*, (25):61–98.

Schellings, T. C. (1978). Micromotives and macrobehaviour. *Norton*.

Snellman, A. and Ekehammar, B. (2005). Ethnic hierarchies, ethnic prejudice, and social dominance orientation. *Journal of Community and Applied Social Psychology*, 15:83–94.

Thomas, K. and Zanetti, G. (2005). *Legge, razza e diritti. La Critical Race Theory negli Stati Uniti*. Diabasis.

Verkuyten, M. and Kinket, B. (2000). Social distances in a multi ethnic society: The ethnic hierarchy among dutch preadolescents. *Social Psychology Quarterly*, 63:75–85.